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Row reduction.

Row reduction – also known as **Gauss-Jordan elimination** — is one of the most important computational tools for dealing with finite dimensional vector spaces. The objects under discussion will be matrices with entries in a field — most often $\mathbb{R}$, but all of this pertains to matrices over *any* field. If A is a matrix, I'll often use A_{ij} or a_{ij} to denote the element of A in the i-th row, j-th column.

Definition. An **elementary row operation** is an operation on the rows of a matrix. It is one of the following:

1. Interchanging two rows of a matrix.
2. Multiplying a row by a nonzero scalar.
3. Adding a multiple of a row to another row.

Here is some useful shorthand. "r_i" will denote the i-th row of the matrix under discussion.

1. $r_i \leftrightarrow r_j$ means: Switch rows i and j.
2. $r_i \to ar_i$ means: Multiply row i by a.
3. $r_i \to r_i + ar_j$ means: Add a times row j to row i. (Row j is left unchanged; row i is replaced by $r_i + ar_j$.)

Remarks.

1. The inverse of an row operation is a row operation.

The inverse of a row operation is an operation which "undoes" the effect of the original operation.
The inverse of $r_i \leftrightarrow r_j$ is $r_i \leftrightarrow r_j$. The inverse of $r_i \to ar_i$ is $r_i \to \frac{1}{a}r_i$. The inverse of $r_i \to r_i + ar_j$ is $r_i \to r_i - ar_j$. $\square$

2. Row operations may be implemented by matrix multiplication.

If E is an elementary row operation, let $E(I)$ denote the matrix obtained by applying E to the $m \times m$ identity matrix. If A is an $m \times n$ matrix, then the matrix product $E(I)A$ is the same as the result of applying E to A.

More specifically, multiplying on the left by

$$\begin{array}{c c} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

interchanges rows i and j. (The schematic above means that the matrix is the $m \times m$ identity matrix, except in those places where elements are shown.)

Multiplying on the left by

$$\begin{array}{c c} & \begin{array}{ccc} & i & \end{array} \\ \begin{array}{c} \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & \\ \cdots & a & \cdots \\ & \vdots & \end{pmatrix} \end{array}$$

multiplies row i by a.

Multiplying on the left by

$$\begin{array}{c c} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & a & \cdots \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

replaces row i with row i plus a times row j. □

Definition. Let A be an $m \times n$ matrix. The **row vectors** of A are the vectors in F^n corresponding to the rows of A. The **row space** of A is the subspace of F^n spanned by the row vectors of A.

Column vectors and the **column space** of A are defined in analogous fashion.

Lemma. If E is an elementary row operation and A is a matrix, then $E(A)$ has the same row space as A.

Proof. If E is an operation of the form $r_i \leftrightarrow r_j$, then $E(A)$ and A have the same rows (except for order), so it's clear that their row vectors have the same span.

If E is an operation of the form $r_i \to ar_i$, then A and $E(A)$ agree except in the i-th row. Since

$$a_1r_1 + \cdots + a_ir_i + \cdots + a_mr_m = a_1r_1 + \cdots + \left(a_i\frac{1}{a}\right)(ar_i) + \cdots + a_mr_m,$$

any element of the row space of A is in the row space of $E(A)$, and vice versa.

Finally, suppose E is a row operation of the form $r_i \to r_i + ar_j$. Then

$$a_1r_1 + \cdots + a_ir_i + \cdots + a_mr_m = a_1r_1 + \cdots + a_i(r_i + ar_j) + \cdots + (a_j - a_ia)r_j + \cdots + a_mr_m,$$

which shows that the row space of A is contained in the row space of $E(A)$.

Conversely,

$$a_1r_1 + \cdots + a_i(r_i + ar_j) + \cdots + a_jr_j + \cdots + a_mr_m = a_1r_1 + \cdots + a_ir_i + \cdots + (a_ia + a_j)r_j + \cdots + a_mr_m,$$

so the row space of $E(A)$ is contained in the row space of A. □

Definition. Two matrices are **row equivalent** if one can be obtained from the other via elementary row operations.

Since the inverse of a row operation is a row operation, if A is row equivalent to B then B is row equivalent to A. It follows that row equivalence is an equivalence relation on the set of $m \times n$ matrices.

Moreover, since row operations preserve row space, row equivalent matrices have the same row space.

Definition. A matrix is **row-reduced** if:

1. The first nonzero element in each row (the **leading coefficient**) is 1. ("First" means reading from left to right along the row. Note that a row can consist entirely of 0's.)

2. A leading coefficient is the only nonzero element in its column.

Proposition. Every matrix is row-equivalent to a row-reduced matrix.

Proof. Let A be an $m \times n$ matrix. Consider the first row of A. If all the elements in row 1 are 0, go to row 2. Otherwise, find the first nonzero element in the row. Suppose it is a_{1j}. Divide row 1 by a_{1j}. Now the leading coefficient in row 1 is 1.

If all the other elements in the j-th column are 0, condition 2 of the definition is satisfied for row 1. Suppose that some element in the j-th column is not 0 — say $a_{ij} \neq 0$. Perform the operation $r_j \to r_j - a_{ij}r_i$.

This does not change row i, and kills a_{ij}. Continuing in this fashion eventually kills all the nonzero elements in column j.

Thus, I've performed row operations which make row 1 satisfy both conditions of the definition.

Now go to row 2. If it's an all-zero row, it satisfies the definition and go on to row 3. Otherwise, find the first nonzero element and divide row 2 by the element to make the leading coefficient 1. Next, use the procedure above to clear out the column containing row 2's leading coefficient. *Note that this doesn't affect row 1's leading cofficient, which is the only nonzero element in its column.*

Continue in this fashion, stopping when you run out of rows. The comment in italics applies to ensure that subsequent operations never "mess up" the leading coefficients that are already fixed. The terminal matrix is row reduced, since each row satisfies the definition. □

Example.

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 7 & 0 \end{bmatrix}$$

is row-reduced.

So is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 7 & 0 \end{bmatrix}.$$

However,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 7 & 0 \end{bmatrix}$$

is not row reduced, because the leading coefficient in row 2 is 2.

Likewise,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 7 & -1 \end{bmatrix}$$

is not row reduced, since the leading coefficient in row 2 is not the only nonzero element in its column. □

Definition. A matrix is in **row reduced echelon form** if:

1. It is row reduced.

2. All the all-zero rows are at the bottom of the matrix.

3. The leading coefficients (which must be 1's) form a "stairstep pattern" from northwest to southeast. Specifically, if the leading coefficients are in the positions $(1, j_1)$, $(2, j_2)$, $\ldots$, (m, j_m), then $j_1 > j_2 > \cdots > j_m$.

Theorem. Every matrix is row equivalent to a row reduced echelon matrix.

Proof. Apply row operations to bring the given matrix to row reduced form. Then observe that a row reduced matrix can be put in row reduced echelon form by simply swapping rows. □

Example. I'm not going to work many of these out by hand, because it is very boring to watch, and because you really need to do this yourself (rather than watching someone else do it).

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -4 & 1 & -3 \\ 2 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -4 & 1 & -3 \\ 0 & -3 & 1 & -5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -\dfrac{1}{4} & \dfrac{3}{4} \\ 0 & -3 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \dfrac{1}{2} & \dfrac{3}{2} \\ 0 & 1 & -\dfrac{1}{4} & \dfrac{3}{4} \\ 0 & -3 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \dfrac{1}{2} & \dfrac{3}{2} \\ 0 & 1 & -\dfrac{1}{4} & \dfrac{3}{4} \\ 0 & 0 & \dfrac{1}{4} & -\dfrac{11}{4} \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & \dfrac{1}{2} & \dfrac{3}{2} \\ 0 & 1 & -\dfrac{1}{4} & \dfrac{3}{4} \\ 0 & 0 & 1 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & -\dfrac{1}{4} & \dfrac{3}{4} \\ 0 & 0 & 1 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -11 \end{bmatrix} \quad \square$$

Next, I'll show that every matrix is row equivalent to a *unique* row reduced echelon matrix.

Lemma. Let $R = \{r_{ij}\}$ be a row reduced echelon matrix with nonzero rows $r_1, \ldots, r_p$. Suppose the leading coefficients of R occur at

$$(1, j_1), (2, j_2), \ldots, \quad \text{where} \quad j_1 < j_2 < \cdots.$$

If $v = \langle v_1, \ldots, v_n \rangle$ and

$$v = a_1 r_1 + \cdots + a_p r_p,$$

then $v_{j_k} = a_k$.

In other words, this lemma tells us about the components of a vector in the row space of R.

Proof.

$$v_{j_k} = \sum_{i=1}^{p} a_i r_{i j_k}.$$

But the only nonzero element in column j_k is $r_{k j_k} = 1$. Therefore, the only nonzero term in the sum is $a_k r_{k j_k} = a_k$. $\square$

Example. Here is a row reduced echelon matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $j_1 = 2$, $j_2 = 4$, and $j_3 = 6$. Consider the following element of the row space:

$$5 \cdot \langle 0, 1, 2, 0, -1, 0 \rangle + (-3) \cdot \langle 0, 0, 0, 1, 2, 0 \rangle + 1 \cdot \langle 0, 0, 0, 0, 0, 1 \rangle = \langle 0, 5, 10, -3, -11, 1 \rangle.$$

Then

$$v_2 = a_1 = 5, \quad v_4 = a_2 = -3, \quad v_6 = a_3 = 1. \ \square$$

Corollary. The nonzero rows of a row reduced echelon matrix are independent.

Proof. Suppose R is a row reduced echelon matrix with nonzero rows $r_1, \ldots, r_p$. Suppose the leading coefficients of R occur at $(1, j_1), (2, j_2), \ldots$, where $j_1 < j_2 < \cdots$. If

$$0 = a_1 r_1 + \cdots + a_p r_p,$$

then the lemma implies that $a_k = v_{j_k} = 0$ for all k. Therefore, the $\{r_i\}$ are independent. □

Corollary. The nonzero rows of a row reduced echelon matrix form a basis for the row space of the matrix.

Proof. The nonzero rows span the row space, and are independent, by the preceding corollary. □

Algorithm. Let V be a finite-dimensional vector space, and let $v_1, \ldots, v_m$ be vectors in V. The object is to find a basis for $W = \langle v_1, \ldots, v_m \rangle$, the subspace spanned by the v_i.

Let M be the matrix whose i-th row is v_i. The row space of M is W. Let R be a row-reduced echelon matrix which is row equivalent to M. Then R and M have the same row space W, and the nonzero rows of R form a basis for W. □

Example. Consider the vectors $v_1 = \langle 1, 0, 1, 1 \rangle$, $v_2 = \langle -2, 1, 1, 0 \rangle$, and $v_3 = \langle 7, -2, 1, 3 \rangle$. I'll find a basis for the subspace $\langle v_1, v_2, v_3 \rangle$. Construct a matrix with the v_i as its rows and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 \\ 7 & -2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors $\langle 1, 0, 1, 1 \rangle$ and $\langle 0, 1, 3, 2 \rangle$ form a basis for $\langle v_1, v_2, v_3 \rangle$. □

Definition. The **dimension** of a vector space V is the cardinality of a basis for V.

The construction in the last example shows that the subspace has **dimension** 2. Note, however, that the definition assumes that you can't have two bases of different cardinalities for a given vector space.

Example. Determine the dimension of the subspace of $\mathbf{R}^3$ spanned by $\langle 1, 2, -1 \rangle$, $\langle 1, 1, 1 \rangle$, and $\langle 2, -2, 1 \rangle$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The subspace has dimension 3, since the row reduced echelon matrix has 3 nonzero rows. □

Definition. The **rank** of a matrix is the dimension of its row space.

Example.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 \\ 7 & -2 & 1 & 3 \end{bmatrix}$$

has rank 2. □

Algorithm. Let V be a finite-dimensional vector space, and let $v_1, \ldots, v_m$ be vectors in V. The object is to determine whether the set $\{v_1, \ldots, v_m\}$ is independent.

Let M be the matrix whose i-th row is v_i. Let R be a row reduced echelon matrix which is row equivalent to M. If R has m nonzero rows, then $\{v_1, \ldots, v_m\}$ is independent. Otherwise, the set is dependent.

If R has p nonzero rows, then R and M have rank p. (They have the same rank, because they have the same row space.) Suppose $p = m$. Since $\{v_i\}$ spans, some subset of $\{v_i\}$ is a basis. However, a basis must contain $p = m$ elements. Therefore, $\{v_i\}$ must be independent.

Any independent subset of the row space must contain $\le p$ elements. Hence, if $m > p$, $\{v_i\}$ must be dependent. $\Box$

Example. The vectors $v_1 = \langle 1, 0, 1, 1\rangle$, $v_2 = \langle -2, 1, 1, 0\rangle$, and $v_3 = \langle 7, -2, 1, 3\rangle$ are dependent:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 \\ 7 & -2 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The row reduced echelon matrix has only two nonzero rows. $\Box$

Proposition. A matrix is row equivalent to a unique row reduced echelon matrix.

Proof. Suppose M row reduces to R, a row reduced echelon matrix with nonzero rows $r_1, \ldots, r_p$. Suppose the leading coefficients of R occur at $(1, j_1), (2, j_2), \ldots$, where $j_1 < j_2 < \cdots$.

Let W be the row space of R and let

$$v = a_1 r_1 + \cdots + a_p r_p$$

be an element of W. In component form, $v = \langle v_1, \ldots, v_n\rangle$.
Claim: The first nonzero component of v must occur in column j_k, for $k = 1, 2, \ldots$.

Suppose a_k is the first a_i which is nonzero. The sum looks like

$$a_k r_k + \cdots + a_p r_p.$$

The first nonzero element of r_k is a 1 at (k, j_k). The first nonzero element in $r_{k+1}, \ldots, r_p$ lies to the right of column j_k. Thus, $v_j = 0$ for $j < j_k$, and $v_{j_k} = a_k$. Evidently, this is the first nonzero component of v. This proves the claim.

This establishes that if a row reduced echelon matrix R' is row equivalent to M, its leading coefficients must lie in the same columns as those of R. For the rows of R' are elements of W, and the claim applies.

Next, I'll show that the nonzero rows of R' are the same as the nonzero row of R.

Consider, for instance, the first nonzero rows of R and R'. Their first nonzero components are 1's lying in column j_1. Moreover, both r_1 and r_1' have zeros in columns j_2, j_3, ...

Suppose $r_1 \ne r_1'$. Then $r_1 - r_1'$ is a nonzero vector in W whose first nonzero component is not in column $j_1, j_2, \ldots \times$.

The same argument applies to show that $r_k = r_k'$ for all k. Therefore, $R = R'$. $\Box$

Example. Elementary row operations correspond to the algebraic operations you normally use to solve systems of linear equations. For example, in the system below I subtract the first equation from the second to eliminate x:

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 2 \\ x & + & 2y & + & z & = & 4 \end{array} \quad \rightarrow \quad \begin{array}{rcrcrcl} x & + & y & - & z & = & 2 \\ & & y & + & 2z & = & 2 \end{array}$$

This corresponds to the row operation $r_2 \rightarrow r_2 - r_1$:

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

Likewise, multiplying or dividing an equation by a number

$$\begin{array}{rcrcrcl} 3x & - & 3y & + & 6z & = & 6 \\ x & - & 2y & + & 5z & = & 7 \end{array} \quad \rightarrow \quad \begin{array}{rcrcrcl} x & - & y & + & 2z & = & 6 \\ x & - & 2y & + & 5z & = & 7 \end{array}$$

corresponds to multiplying or dividing a row of a matrix by a number:

$$\begin{bmatrix} 3 & -3 & 6 & 6 \\ 1 & -2 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 1 & -2 & 5 & 7 \end{bmatrix}$$

(Swapping two rows of a matrix corresponds to writing two equations in the opposite order, which is normally not a useful algebraic operation.) $\square$

Definition. Let

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & 0 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

be a system of linear equations.

The corresponding matrix equation is

$$A \cdot x = 0, \quad \text{where} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The **solution space** for the system is the set of all vectors x which satisfy the equation. An element x in the solution space is a **solution vector**.

As usual, the coefficients of the matrix and the components of the solution vector are assumed to lie in a base field F.

It's easy to see that the solution space is a subspace of F^n. First, since $A \cdot 0 = 0$, the solution space is nonempty. Next, suppose that u and v are in the solution space of $Ax = 0$, and let $a \in F$. Then

$$A(au + v) = aA \cdot u + A \cdot v = a \cdot 0 + 0 = 0.$$

This shows that a linear combination of solutions is a solution. Hence, the solution space is a subspace.

Theorem. If A and B are row equivalent $m \times n$ matrices, then the homogeneous systems

$$AX = 0 \quad \text{and} \quad BX = 0$$

have the same solution spaces.

Here X denotes the column vector $\langle x_1, \ldots, x_n \rangle$.

Proof. If A and B are row equivalent, there is a sequence of elementary row operations which convert one to the other. Let $E_1, \ldots, E_p$ be such a sequence of row operations, represented as matrices. Then

$$E_1 \cdots E_p A = B.$$

If $Ax = 0$, then $Bx = E_1 \cdots E_p Ax = 0$. Therefore, the solution space of A is contained in the solution space of B.

Since the inverse of an elementary row operation is a row operation, there are also find elementary row operations with matrices $F_1, \ldots, F_q$ such that

$$F_1 \cdots F_q B = A.$$

(In fact, the F's are the inverses of the matrices $E_1, \ldots, E_p$, but that fact isn't required for the proof.)

If $Bx = 0$, then $Ax = F_1 \cdots F_q Bx = 0$. Therefore, the solution space of B is contained in the solution space of A.

Hence, A and B have the same solution space. □

In particular, you can solve a system $Ax = 0$ by reducing A to a row reduced echelon matrix R. By the theorem, $Rx = 0$ has the same solution space as $Ax = 0$. It is easy to read off a parametrized solution from a row reduced echelon system.

Example. Suppose you're given a homogeneous system $Ax = 0$, and A row reduces to

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 5 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose that the variables in the original system were a, b, c, d, e, and f. The matrix above yields the equations

$$\begin{array}{ccccccccccc} a & & & + & 2c & + & 3d & & & + & 5f & = & 0 \\ & & b & + & c & - & d & & & + & 2f & = & 0 \\ & & & & & & & & e & - & 2f & = & 0 \end{array}$$

Translating the matrix into equation form,

$$\begin{aligned} a &= -2c - 3d - 5f, \\ b &= c + d - 2f, \\ e &= 2f. \end{aligned}$$

(This is the parametrized form of the solution to the system.)

Hence, a general element of the solution space has the form

$$\begin{bmatrix} -2c - 3d - 5f \\ -c + d - 2f \\ c \\ d \\ 2f \\ f \end{bmatrix} = c \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + f \cdot \begin{bmatrix} -5 \\ -2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Hence,

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space (which therefore has dimension 3). □

Here is a summary of the previous results. Let A be an $m \times n$ matrix. To solve $Ax = 0$, row reduce A to a row reduced echelon matrix R. If R has p nonzero rows, it has p leading coefficients.

The p leading coefficients correspond to the dependent variables in the solution to the system. The other $n - p$ variables are **free variables** or **parameters**. The solution space will have dimension $n - p$.

As a special case, if there are n leading coefficients, the solution space is trivial — the only solution to $Ax = 0$ is $x = 0$.