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Normal subgroups.

Let G be a group, $H < G$. In general, G/H is just a set. It is tempting, however, to try to define a binary operation on cosets by

$$(aH) \cdot (bH) = (ab)H.$$

The idea is to use this binary operation to make G/H into a group. However, a problem arises with the definition of the operation. The cosets aH and bH above could be written in terms of other elements (**coset representative**). Suppose that $aH = a'H$ and $bH = b'H$ for some other elements a', b'. Then

$$(a'H) \cdot (b'H) = (a'b')H.$$

The product will not make sense — it won't be **well-defined** — unless $(ab)H = (a'b')H$. This will not be the case for an arbitrary subgroup $H < G$.

I'm going to play around to see what sort of conditions I might put on H to make things work. First, an easy lemma.

Lemma. Let G be a group, $H < G$. $aH = H$ if and only if $a \in H$. (The same thing works for right cosets.)

Proof. ($\Rightarrow$) Suppose $aH = H$. $1 \in H$, so $a \cdot 1 \in a \cdot H = H$, i.e. $a \in H$.

($\Leftarrow$) Suppose $a \in H$. Then $aH \subset H$ by closure. But $a^{-1} \in H$, so $a^{-1} \subset H$ by closure. Hence, $H = aa^{-1}H \subset aH$, and therefore $aH = H$. $\square$

Now if aH is a coset and $h \in H$, then $aH = ahH$. (This has the effect of varying the representative of the coset aH — i.e. ah here is like a' above.) I want

$$(aH)(bH) = (ahH)(bH), \text{ that is, } (ab)H = (ahb)H.$$

Moreover, this equation must be true for any element $b \in G$. Since $ab \cdot 1 \in (ab)H$, $ab \in (ahb)H$, or $ab = ahbh_0$, for some $h_0 \in H$. Again, this equation must hold for all $b \in G$ and all $h \in H$. Simplify and rearrange:

$$b = hbh_0, \text{ so } h_0^{-1} = b^{-1}hb.$$

In other words, for all $b \in G$ and all $h \in H$,

$$b^{-1}hb = h_0^{-1} \in H.$$

In terms of sets, for all $b \in G$,

$$b^{-1}Hb \subset H.$$

Definition. A subgroup $H < G$ is **normal** if

$$gHg^{-1} \subset H \quad \text{for all } g \in G.$$

(Since the statement runs over *all* $g \in G$, I could just as well say "$g^{-1}Hg \subset H$".) Write $H \triangleleft G$ to mean that H is a normal subgroup of G.

It does no harm to replace the inclusion with equality.

Lemma. $H \triangleleft G$ if and only if $gHg^{-1} = H$ for all $g \in G$.

Proof. ($\Leftarrow$) If $gHg^{-1} = H$ for all $g \in G$, then *a fortiori* $gHg^{-1} \subset H$ for all $g \in G$. Therefore, $H \triangleleft G$.

($\Rightarrow$) Suppose $H \triangleleft G$. Let $g \in G$. Then $gHg^{-1} \subset H$. On the other hand, using g^{-1} in place of g, I obtain $g^{-1}Hg \subset H$. Multiply this inclusion on the left by g and on the right by g^{-1} to obtain $H \subset gHg^{-1}$. Therefore, $gHg^{-1} = H$. $\square$

If you *know* a subgroup is normal, you use the equation form because it is "stronger". On the other hand, if you're trying to *prove* that a subgroup is normal, you use the inclusion form because there is less to prove!

Now I'll show that the definition of normality does what I wanted it to.

Theorem. Let G be a group, $H < G$. The following statements are equivalent:

1. $H \triangleleft G$
2. For all $g \in G$, $gH = Hg$. (Thus, every left coset is a right coset and every right coset is a left coset.)
3. Coset multiplication is well-defined.

By the third statement, I mean that if $a_1 H = a_2 H$ and $b_1 H = b_2 H$, then $a_1 b_1 H = a_2 b_2 H$. *When* I have a well-defined multiplication, I will define multiplication of cosets by $(aH)(bH) = abH$.

Proof. $(1 \Rightarrow 2)$ If $H \triangleleft G$ and $g \in G$, then $gHg^{-1} = H$, so $gHg^{-1}g = Hg$, or $gH = Hg$.

$(2 \Rightarrow 3)$ Suppose $gH = Hg$ for all $g \in G$. Suppose

$$a_1 H = a_2 H \text{ and } b_1 H = b_2 H.$$

Then

$$a_1 b_1 H = a_1 b_2 H = a_1 H b_2 = a_2 H b_2 = a_2 b_2 H.$$

$(3 \Rightarrow 1)$ Suppose coset multiplication is well defined. I want to show $H \triangleleft G$. Let $g \in G$. I will show $gHg^{-1} \subset H$.

Do it with elements. Let $h \in H$. I will show $ghg^{-1} \in H$.

By an earlier lemma, $hH = H$, and surely $gH = gH$, so (since coset multiplication is well-defined) $ghH = gH$. Again, $g^{-1}H = g^{-1}H$, so $(gh)g^{-1}H = gg^{-1}H = 1 \cdot H = H$. By another lemma, this shows that $ghg^{-1} \in H$. Therefore, $H \triangleleft G$. $\square$

The point of all this way to make the set of cosets G/H into a group via coset multiplication.

Proposition. If $H \triangleleft G$, the set of left cosets G/H becomes a group under coset multiplication.

Proof. *Associativity:*

$$(aH \cdot bH) \cdot cH = (ab)H \cdot cH = (abc)H, \quad aH \cdot (bH \cdot cH) = aH \cdot (bc)H = (abc)H.$$

Identity: $1H \cdot aH = aH = aH \cdot 1H$ for all $a \in G$.

Inverses: $aH \cdot a^{-1}H = 1H = a^{-1} \cdot aH$ for all $a \in G$. $\square$

Examples. 0. If G is a group, $\{1\}$ and G are normal in G. $\square$

1. If G is abelian, every subgroup is normal. For if $g \in G$, then $gHg^{-1} = Hgg^{-1} = H$. $\square$

2. In D_3, the rotation subgroup is normal. The subgroup consisting of the identity and any reflection is not normal. $\square$

3. If G is a group and $H < G$, $|G/H|$ is called the **index** of H in G, and is denote $(G : H)$.

A subgroup of index 2 is always normal. For suppose $(G : H) = 2$. This means that H has two left cosets and two right cosets. (Why are the same number of left and right cosets? See the next lemma.) One coset is always H itself. Take $g \notin H$. Then gH is the other left coset, Hg is the other right coset, and

$$H \cup gH = G = H \cup Hg.$$

But these are disjoint unions, so I can cancel the H's and obtain $gH = Hg$. By an earlier result, this means that H is normal. □

Here is the lemma I used in the last example.

Lemma. Let G be a group, $H < G$. The set of left cosets G/H may be put in 1-1 correspondence with the set of right coset $H\backslash G$.

Proof. Define $\phi : G/H \to H\backslash G$ by $\phi(gH) = Hg^{-1}$. I need to show ϕ is well-defined.

Suppose $g_1H = g_2H$. Then $g_1 = g_1 \cdot 1 \in g_1H = g_2H$, so $g_1 = g_2h$ for some $h \in H$. Then

$$\phi(g_1H) = Hg_1^{-1} = H(g_2h)^{-1} = Hh^{-1}g_2^{-1} = Hg_2^{-1} = \phi)g_2H).$$

Next, define $\psi : H\backslash G \to G/H$ by $\psi(Hg) = g^{-1}H$. A computation similar to the one I just did shows ψ is well-defined. ϕ and ψ are inverses, so either one gives a bijection of G/H with $H\backslash G$. □

It is natural to see how a new construction interacts with things like unions and intersections. Since the union of subgroups is not a subgroup in general, it's unreasonable to expect a union of normal subgroups to be a normal subgroup. However, intersections work properly.

Lemma. The intersection of a family of normal subgroups is a normal subgroup.

Proof. Let G be a group, and let $\{H_a\}_{a\in A}$ be a family of normal subgroups of G. Let $H = \cap_{a\in A} H_a$. I want to show that $H \lhd G$. Since the intersection of a family of subgroups is a subgroup, it remains to show that H is normal.

Let $g \in G$ and let $h \in H$. I must show $ghg^{-1} \in H$. Now $h \in H$ implies $h \in H_a$ for all a, so (since $H_a \lhd G$ for all a) $ghg^{-1} \in H_a$ for all a. Therefore, $ghg^{-1} \in \cap_{a\in A} H_a = H$. Therefore, H is normal. □

Definition. Let G be a group, and let $S \subset G$. The intersection of all normal subgroups of G containing S is the **normal subgroup generated by** S.

Example. Here's D_3 again. I keep using D_3 because you can show that groups of order < 6 are all abelian, and subgroups of abelian groups are automatically normal. So I have to go up to order 6 (at least) to get examples of non-normal subgroups.

	id	r_1	r_2	m_1	m_2	m_3
id	id	r_1	r_2	m_1	m_2	m_3
r_1	r_1	r_2	id	m_3	m_1	m_2
r_2	r_2	id	r_1	m_2	m_3	m_1
m_1	m_1	m_2	m_3	id	r_1	r_2
m_2	m_2	m_3	m_1	r_2	id	r_1
m_3	m_3	m_1	m_2	r_1	r_2	id

The rotation subgroup $H = \{id, r_1, r_2\}$ is a normal subgroup of D_3. In this case, D_3/H is a group with two elements:

$$D_3/H = \{H = \{id, \rho_1, \rho_2\}, m_1H = \{m_1, m_2, m_3\}\}.$$

Note that $\mu_1H = \mu_2H = \mu_3H$ — these are all the same coset.

Here is the group table for D_3/H:

	H	m_1H
H	H	m_1H
m_1H	m_1H	H

Up to notation, this is “the” group of order 2.

More generally, consider the group D_{2n} of symmetries of the regular n-gon. This group has a subgroup of rotations H consisting of rotations through the angles $\dfrac{2\pi k}{n}$, where $0 \le k < n$. This subgroup is normal, since it has index 2. To see this directly, observe that if ρ is a rotation and τ is also a rotation, $\tau\rho\tau^{-1}$ is obviously a rotation. On the other hand, suppose τ is a reflection. Then $\tau\rho\tau^{-1}$ is orientation-preserving, so it must also be a rotation. □