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Functions.

If X and Y are sets, $f : X \to Y$ denotes a function from X to Y. The set X of possible inputs is called the **domain** of f. The set $f(X)$ of possible outputs is called the **image** (or **range**) of f.

(I'll often use the words *function* and *map* interchangeably.)

For example, let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. The domain is $\mathbb{R}$ and the range is the set of nonnegative real numbers $\{y \mid y \ge 0\}$.

Definition. Let $f : X \to Y$.

1. f is **injective** (or **one-to-one**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Equivalently, if $x_1 \ne x_2$, then $f(x_1) \ne f(x_2)$ — different inputs produce different outputs.

2. f is **surjective** (or **onto**) if for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$. That is, every $y \in Y$ is the target of some $x \in X$.

3. f is **bijective** (or a **one-to-one correspondence**) if f is injective and surjective.

An injective map is called an **injection**, a surjective map is called a **surjection**, and a bijective map is called a **bijection**.

Warning! There is a difference between a one-to-one map (an injective map) and a one-to-one correspondence (a bijective map).

Examples. 1. Properly speaking, a function is not just a rule matching inputs to outputs. You need to specify the allowable inputs and the allowable output as part of the definition.

Consider the function $f(x) = x^2$ on $\mathbb{R}$.

If $f : \mathbb{R} \to \mathbb{R}$, f is not injective (since, e.g., $f(2) = f(-2) = 4$) and f is not surjective (since, e.g., there is no $x \in \mathbb{R}$ such that $f(x) = -1$).

If $f : \mathbb{R}^+ \to \mathbb{R}$ (where $\mathbb{R}^+$ denotes the positive reals), then f is injective but not surjective.

If $f : \mathbb{R} \to \mathbb{R}^+$, then f is surjective but not injective.

If $f : \mathbb{R}^+ \to \mathbb{R}^+$, then f is bijective. $\square$

2. Sets X and Y have the **same cardinality** if there is a bijection $f : X \to Y$. Cardinality is a measure of "how many". For example, there is a bijection $f : \mathbb{Z} \to 2\mathbb{Z}$ given by $f(n) = 2n$, where $2\mathbb{Z}$ denotes the set of even integers. In this sense, there are "as many" even integers as there are integers.

On the other hand, there is no bijection from $\mathbb{Z}$ to $\mathbb{R}$. Both sets are infinite, but $\mathbb{R}$ is strictly "bigger". $\square$

Definition. Let $f : X \to Y$. $g : Y \to X$ is the **inverse** of f if

$$f(g(y)) = y \text{ for all } y \in Y, \quad \text{and} \quad g(f(x)) = x \text{ for all } x \in X.$$

The inverse of f is, roughly, a function which "undoes" the effect of f.

The word "the" in the definition is justified for the following reason. Suppose that $g_1, g_2 : Y \to X$ are inverses for f. Then for all $y \in Y$,

$$f(g_1(y)) = y = f(g_2(y)) \quad \text{so} \quad g_1(f(g_1(y))) = g_1(f(g_2(y))).$$

Since $g_1(f(\text{junk})) = \text{junk}$, this means that $g_1(y) = g_2(y)$. Since this is true for all $y \in Y$, $g_1 = g_2$. In other words, if f has an inverse, the inverse is unique.

If f has an inverse, it is denoted f^{-1}. (This does not mean $\dfrac{1}{f}$!)

Of course, not all functions have inverses. Suppose, for example, that $f(x) = x^2$, where $f : \mathbb{R} \to \mathbb{R}$. To "undo" the effect of f, I have to be able to take an output — say 4 — and tell what input produced it. However, all I can say is that 4 comes from either 2 or -2. Worse yet, there is no input which produces -1 as an output.

Two things have gone wrong here. First, different inputs can produce different outputs — f is not one-to-one. Second, some outputs are not produced by any inputs — f is not onto. The following (extremely useful) lemma shows that these are, in a sense, the only two things that can go wrong if you're looking for an inverse.

Lemma. Let $f : X \to Y$. f has an inverse if and only if f is bijective.

Proof. I have two things to show. First, I must show that *if* $f : X \to Y$ has an inverse, *then* f is bijective. Second, I must show that *if* f is bijective, *then* f has an inverse.

Suppose first that $f : X \to Y$ has an inverse. I need to show that f is bijective, i.e. that f is injective and that f is surjective.

First, I'll show f is injective. Suppose $f(x_1) = f(x_2)$, where $x_1, x_2 \in X$. Apply the inverse f^{-1} to both sides: $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$, so $x_1 = x_2$. Therefore, f is injective.

Next, let $y \in Y$. I need to find an $x \in X$ such that $f(x) = y$. But $f(f^{-1}(y)) = y$, so I can take x to be $f^{-1}(y)$. Thus, f is surjective.

Since f is injective and surjective, it's bijective.

Now assume f is bijective. I must show that f has an inverse. This means that I have to construct a function $g : Y \to X$ which undoes the effect of f. Let $y \in Y$. How can I define $g(y)$? Well, f is surjective, so there is an $x \in X$ such that $f(x) = y$. Moreover, f is injective, so the x that f maps to y is unique. Hence, I can define $g(y) = x$. Now I want to show that g is the inverse of f.

First, let $x \in X$. What is $g(f(x))$? By definition of g, I look for the element of X which f maps to $f(x)$, and that element will be $g(f(x))$. Obviously, the element x maps to $f(x)$, so $g(f(x)) = x$.

Next, let $y \in Y$. What is $f(g(y))$? Well, $g(y) = x$, where $f(x) = y$. So $f(g(y)) = f(x) = y$.

Thus, f and g are inverses. $\square$

This lemma is used in both directions. On the one hand, if you need to show two sets X and Y are in one-to-one correspondence, the lemma suggests that you construct maps $f : X \to Y$ and $g : Y \to X$ which are inverses.

On the other hand, suppose you need to show a function is invertible. The lemma says that you can do this by showing that the function is one-to-one and onto.