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Variation of parameters.

Variation of parameters constructs the general solution to a nonhomgeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

given a pair of independent solutions y_1, y_2 to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The idea is similar to that used in **reduction of order**. Guess a particular solution of the form

$$y = u_1 y_1 + u_2 y_2.$$

The problem is to determine u_1 and u_2.
Plug y into the differential equation and attempt to solve for u_1 and u_2. The derivatives are

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

$$y'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$

Substitute:

$$f(x) = y'' + py' + qy =$$

$$\begin{array}{ccccccccccc} u_1'' y_1 & + & 2u_1' y_1' & + & u_1 y_1'' & + & u_2'' y_2 & + & 2u_2' y_2' & + & u_2 y_2'' \\ & + & pu_1' y_1 & + & pu_1 y_1' & & & + & pu_2' y_2 & + & pu_2 y_2' \\ & & & + & qu_1 y_1 & & & & & + & qu_2 y_2 \\ u_1'' y_1 & + & (2u_1' y_1' + pu_1' y_1) & & & + & u_2'' y_2 & + & (2u_2' y_2' + pu_2' y_2) & & \end{array}$$

($y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$ since y_1 and y_2 are homogeneous solutions.)
The disgusting mess that remains can be rearranged to obtain

$$[(u_1'' y_1 + u_2'' y_2) + (u_1' y_1' + u_2' y_2')] + (u_1' y_1' + u_2' y_2') + p(u_1' y_1 + u_2' y_2) =$$

$$(u_1' y_1 + u_2' y_2)' + (u_1' y_1' + u_2' y_2') + p(u_1' y_1 + u_2' y_2).$$

So

$$(u_1' y_1' + u_2' y_2')' + (u_1' y_1' + u_2' y_2') + p(u_1' y_1 + u_2' y_2) = f(x).$$

Note that this is *one* equation in the *two* unknowns u_1 and u_2. So I have one degree of freedom, and I can use it to set up a second equation which I can use together with the equation above. Since the expression $u_1' y_1' + u_2' y_2'$ appears twice, I will take as my second equation

$$u_1' y_1 + u_2' y_2 = 0. \tag{1}$$

The equation above then simplies to

$$u_1' y_1' + u_2' y_2' = f(x). \tag{2}$$

I want to solve equations (1) and (2) simultaneously. I can rewrite them in matrix form as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}.$$

I'll refer to this as the **variation of parameters** equation. Notice that the determinant of the matrix on the left is the Wronskian of y_1 and y_2, which makes it a bit easier to remember. *Rather than repeat the* *computations above, I will start problems with the equation above.* This seems to be a reasonable compromise between memorization and computation.

One other thing: I'll need the formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example. $(D^2 - 3D + 2)y = \dfrac{1}{1 + e^{-x}}$

Since $\dfrac{1}{1 + e^{-x}}$ is not a solution to a linear constant coefficient homogeneous equation, you can't use undetermined coefficients.

The homogeneous equation $(D^2 - 3D + 2)y = 0$ has fundamental solutions $y_1 = e^x$ and $y_2 = e^{2x}$.

The particular solution is

$$y = u_1 e^x + u_2 e^{2x}.$$

(Actually, by including arbitrary constants as I integrate, I can use this to get the general solution.)

The variation of parameters equation is

$$\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{1 + e^{-x}} \end{bmatrix}.$$

Invert the coefficient matrix on the left and put the inverse on the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{-3x} \begin{bmatrix} 2e^{2x} & -e^{2x} \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{1}{1 + e^{-x}} \end{bmatrix} = \begin{bmatrix} e^{-3x} \dfrac{-e^{2x}}{1 + e^{-x}} e^{-3x} \dfrac{e^x}{1 + e^{-x}} \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = e^{-3x} \frac{-e^{2x}}{1 + e^{-x}} = \frac{-e^{-x}}{1 + e^{-x}}, \quad u_1 = \ln\left(1 + e^{-x}\right) + c_1,$$

$$u_2' = e^{-3x} \frac{e^x}{1 + e^{-x}} = \frac{e^{-2x}}{1 + e^{-x}}, \quad u_2 = -e^{-x} + \ln\left(1 + e^{-x}\right) + c_2.$$

Here's the work for the integral. Let $w = e^{-x}$, so $dw = -e^{-x}\,dx$:

$$\int \frac{e^{-2x}}{1 + e^{-x}}\,dx = -\int \frac{w}{1 + w}\,dw = -\int \left(1 - \frac{1}{1 + w}\right) =$$

$$-w + \ln|1 + w| + C = -e^{-x} + \ln\left(1 + e^{-x}\right) + C.$$

As an alternative, you can omit the arbitrary constants in u_1 and u_2, obtain a particular solution, and add it to the homogeneous solution to get the general solution.

The general solution is

$$y = c_1 e^x + c_2 e^{2x} + \left(e^x + e^{2x}\right) \ln\left(1 + e^{-x}\right). \ \square$$

Example. Solve $(2x - 1)y'' - 4xy' + 4y = (2x - 1)^2 e^{-x}$, if $y_1 = x$ and $y_2 = e^{2x}$ are solutions to the corresponding homogeneous equation.

(In cases where it's not reasonable to expect you to find two homogeneous solutions yourself, they will be given.)

First, I need to put the equation into standard form:

$$y'' - \frac{4x}{2x-1}y' + \frac{4}{2x-1}y = (2x-1)e^{-x}.$$

The variation of parameters equation is

$$\begin{bmatrix} x & e^{2x} \\ 1 & 2e^{2x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \begin{bmatrix} 0 \\ (2x-1)e^{-x} \end{bmatrix}.$$

Invert the coefficient matrix and put it on the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{(2x-1)e^{2x}} \begin{bmatrix} 2e^{2x} & -e^{2x} \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ (2x-1)e^{-x} \end{bmatrix} = \begin{bmatrix} -e^{-x} \\ xe^{-3x} \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = -e^{-x}, \quad u_1 = e^{-x} + c_1,$$

$$u_2' = xe^{-3x}, \quad u_2 = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c_2.$$

The general solution is

$$y = \frac{2}{3}xe^{-x} - \frac{1}{9}e^{-x} + c_1 x + c_2 e^{2x}. \square$$

Example. $y'' + y = 3\sec x - x^2 + 1$

I'll solve these two equations separately:

$$y'' + y = 3\sec x \quad \text{and} \quad y'' + y = -x^2 + 1.$$

I'll do the first by variation of parameters, the second by undetermined coefficients. Then I'll add the results (**superposition**) to obtain the solution to the original equation.

First, the homogeneous equation $y''+y = 0$ is $(D^2+1)y = 0$. The solutions are $y_1 = \cos x$ and $y_2 = \sin x$.

For $y'' + y = 3\sec x$, the solution should have the form

$$y_p = u_1 \cos x + u_2 \sin x.$$

The variation of parameters equation is

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 3\sec x \end{bmatrix}.$$

Invert the coefficient matrix and put it in the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ 3\sec x \end{bmatrix} = \begin{bmatrix} -3\dfrac{\sin x}{\cos x} \\ 3 \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = -3\frac{\sin x}{\cos x}, \quad u_1 = 3\ln|\cos x|,$$

$$u_2' = 3, \quad u_2 = 3x.$$

Thus,

$$y_p = 3\cos x \ln|\cos x| + 3x\sin x.$$

Next, consider $(D^2+1)y_p = -x^2+1$. x^2 and 1 are killed by D^3, so apply D^3 to both sides:

$$D^3(D^2+1)y_p = 0, \quad y_p = A + Bx + Cx^2 + D\cos x + E\sin x.$$

$\cos x$ and $\sin x$ are homogeneous solutions. Throw them out, leaving

$$y_p = A + Bx + Cx^2.$$

Differentiate:

$$y_p' = B + 2Cx, \quad y_p'' = 2C.$$

Therefore,

$$y_p'' + y_p = 2C + (A + Bx + Cx^2) = -x^2 + 1.$$

Equate corresponding coefficients:

$$\begin{array}{ccccccc} A & + & & & 2C & = & 1, \\ & & B & & & = & 0, \\ & & & & C & = & -1. \end{array}$$

Solve simultaneously: $A = 3$, $B = 0$, $C = -1$. Thus,

$$y_p = 3 - x^2.$$

The general solution to the original equation is

$$y = c_1\cos x + c_2\sin x + 3\cos x\ln|\cos x| + 3x\sin x + 3 - x^2. \ \square$$