3-25-1994

Linear systems with constant coefficients.

Here is a system of n equations in n unknowns:

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n, \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n. \end{aligned}$$

This is a **constant coefficient linear homogeneous system**. Thus, the coefficients a_{ij} are constant, and you can see that the equations are linear in the variables $x_1, \ldots, x_n$ and their derivatives. The reason for the term "homogeneous" will be clear when I've written the system in matrix form.

The primes on $x_1', \ldots, x_n'$ denote differentiation with respect to an independent variable t. The problem is to solve for $x_1, \ldots, x_n$ in terms of t.

Write the system in matrix form as

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Equivalently,

$$\vec{x}' = A\vec{x}.$$

(A **nonhomogeneous system** would look like $\vec{x}' = A\vec{x} + \vec{b}$.)

It's possible to solve such a system if you know the eigenvalues (and possibly the eigenvectors) for the coefficient matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

First, I'll do an example which shows that you can solve small linear systems by brute force.

Example. Consider the system of differential equations

$$\frac{dx_1}{dt} = x_1' = -29x_1 - 48x_2,$$

$$\frac{dx_2}{dt} = x_2' = 16x_1 + 27x_2.$$

The idea is to solve for x_1 and x_2 in terms of t.

One approach is to use brute force. Solve the first equation for x_2, then differentiate to find x_2':

$$x_2 = \frac{1}{48}(-x_1' - 29x_1), \quad x_2' = \frac{1}{48}(-x_1'' - 29x_1').$$

Plug these into second equation:

$$\frac{1}{48}(-x_1'' - 29x_1') = 16x_1 + 27 \cdot \frac{1}{48}(-x_1' - 29x_1), \quad x_1'' + 2x_1' - 15x_1 = 0.$$

This is a constant coefficient linear homogeneous equation in x_1. The characteristic equation is $m^2 - 2m - 15 = 0$. The roots are $m = -5$ and $m = 3$. Therefore,

$$x_1 = c_1 e^{-5t} + c_2 e^{3t}.$$

Plug back in to find x_2:

$$x_2 = \frac{1}{48}(-x_1' - 29x_1) = \frac{1}{48}\left(-(-5c_1 e^{-5t} + 3c_2 e^{3t}) - 29(c_1 e^{-5t} + c_2 e^{3t})\right) = -\frac{1}{2}c_1 e^{-5t} - \frac{2}{3}c_2 e^{3t}. \square$$

The procedure works, but it's clear that the computations would be pretty horrible for larger systems. To describe a better approach, look at the coefficient matrix:

$$A = \begin{bmatrix} -29 & -48 \\ 16 & 27 \end{bmatrix}$$

Find the eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -29 - \lambda & -48 \\ 16 & 27 - \lambda \end{vmatrix} = (\lambda + 29)(\lambda - 27) + 16 \cdot 48 = \lambda^2 + 2\lambda - 15.$$

This is the same polynomial that appeared in the example. Since $\lambda^2 + 2\lambda - 15 = (\lambda + 5)(\lambda - 3)$, the eigenvalues are $\lambda = -5$ and $\lambda = 3$.

Thus, you don't need to go through the process of eliminating x_2 and isolating x_1. You know that

$$x_1 = c_1 e^{-5t} + c_2 e^{3t}$$

once you know the eigenvalues of the coefficient matrix. You can now finish the problem as above by plugging x_1 back in to solve for x_2.

This is better than brute force, but it's still cumbersome if the system has more than two variables.

I can improve things further by making use of eigenvectors as well as eigenvalues. Consider the system

$$\vec{x}' = A\vec{x}.$$

Suppose λ is an eigenvalue of A with eigenvector v. This means that

$$Av = \lambda v.$$

I claim that $\vec{x} = ce^{\lambda t}v$ is a solution to the equation, where c is a constant. To see this, plug it in:

$$\vec{x}' = c\lambda e^{\lambda t} v = ce^{\lambda t}(\lambda v) = ce^{\lambda t}(Av) = A(ce^{\lambda t}v) = A\vec{x}.$$

To obtain the *general solution* to $\vec{x}' = A\vec{x}$, you should have "one arbitrary constant for each differentiation". In this case, you'd expect n arbitrary constants. This discussion should make the following result plausible.

> Suppose the matrix A has n independent eigenvectors $v_1, \ldots, v_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the general solution to $\vec{x}' = A\vec{x}$ is
>
> $$\vec{x} = c_1 e^{\lambda_1 t} v_1 + \cdots c_n e^{\lambda_n t} v_n.$$

Example. Solve

$$\frac{dx_1}{dt} = x_1' = -29x_1 - 48x_2,$$

$$\frac{dx_2}{dt} = x_2' = 16x_1 + 27x_2.$$

The matrix form is

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -29 & -48 \\ 16 & 27 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The matrix

$$A = \begin{bmatrix} -29 & -48 \\ 16 & 27 \end{bmatrix}$$

has eigenvalues $\lambda = -5$ and $\lambda = 3$. I need to find the eigenvectors.

Consider $\lambda = -5$:

$$A + 5I = \begin{bmatrix} -24 & -48 \\ 16 & 32 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a + 2b = 0$, or $a = -2b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2b \\ b \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Take $b = 1$. The eigenvector is $\langle -2, 1 \rangle$.

Now consider $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -32 & -48 \\ 16 & 24 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a + \frac{3}{2}b = 0$, or $a = -\frac{3}{2}b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}b \\ b \end{bmatrix} = b \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}.$$

Take $b = 2$. The eigenvector is $\langle -3, 2 \rangle$.

You can check that the vectors $\langle -2, 1 \rangle$, $\langle -3, 2 \rangle$, are independent. Hence, the solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \quad \square$$

Remarks.

1. One can show that eigenvectors corresponding to different eigenvectors are independent. That is, if

$$\begin{aligned} &v_1 \text{ is an eigenvector for } \lambda_1, \\ &v_2 \text{ is an eigenvector for } \lambda_2, \\ &\vdots \\ &v_n \text{ is an eigenvector for } \lambda_n, \end{aligned}$$

and the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, then $\{v_1, \ldots, v_n\}$ is independent.

2. It follows that an $n \times n$ matrix A is guaranteed to have n independent eigenvectors if it has n distinct eigenvalues. In this case, the method above can be used to solve the corresponding differential equation. It's also clear that problems may occur if the characteristic polynomial has *repeated roots.*

3. A square matrix A is **symmetric** if $A_{ij} = A_{ji}$, where A_{ij} denotes the element in row i, column j. Thus, a symmetric matrix is symmetric across its main diagonal. For example,

$$\begin{bmatrix} 1 & -2 & 9 \\ -2 & 7 & -3 \\ 9 & -3 & 11 \end{bmatrix}$$

is symmetric. A theorem of linear algebra (the Principal Axis theorem) says that a symmetric $n \times n$ matrix has n independent eigenvectors. It follows that a system with symmetric coefficient matrix can be solved by the eigenvector method.

Later on, I'll discuss how to handle the repeated root case, as well as the case where there aren't enough eigenvectors. For the moment, I'll only consider the case of an $n \times n$ matrix with n independent eigenvectors.

Example. (complex roots) Solve

$$x_1' = 5x_1 + 5x_2,$$
$$x_2' = -4x_1 - 3x_2.$$

The characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & 5 \\ -4 & -3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5.$$

The eigenvalues are $\lambda = 1 \pm 2i$. You can check that the eigenvectors are:

$$\lambda = 1 - 2i: \quad b \cdot \begin{bmatrix} -2+i \\ 2 \end{bmatrix}$$
$$\lambda = 1 + 2i: \quad b \cdot \begin{bmatrix} -2-i \\ 2 \end{bmatrix}$$

(Observe that the eigenvectors are conjugates of one another. In fact, this is always true when you have a complex eigenvalue.)

I'll discuss the complex eigenvalue case later — so if you don't see how to get these vectors, just take them for granted for the moment.

The eigenvector method gives the following *complex* solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{(1-2i)t} \begin{bmatrix} -2+i \\ 2 \end{bmatrix} + c_2 e^{(1+2i)t} \begin{bmatrix} -2-i \\ 2 \end{bmatrix} =$$

$$e^t \begin{bmatrix} (-2(c_1+c_2) + i(c_1-c_2)) \cos 2t + ((c_1+c_2) + 2i(c_1-c_2)) \sin 2t \\ 2(c_1+c_2)\cos 2t - 2i(c_1-c_2)\sin 2t \end{bmatrix}.$$

Note that the constants occur in the combinations $c_1 + c_2$ and $i(c_1 - c_2)$. Something like this will *always* happen in the complex case. Set $d_1 = c_1 + c_2$ and $d_2 = i(c_1 - c_2)$. The solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^t \begin{bmatrix} (-2d_1 + d_2)\cos 2t + (d_1 + 2d_2)\sin 2t \\ 2d_1 \cos 2t - 2d_2 \sin 2t \end{bmatrix}.$$

In fact, if you're given initial conditions for x_1 and x_2, the new constants d_1 and d_2 will turn out to be real numbers. $\square$

Example. (A compartment model) Two tanks hold 50 gallons of liquid each. The first tank starts with 25 pounds of dissolved salt, while the second starts with pure water. Pure water flows into the first tank at 3 gallons per minute; the well-stirred micture flows into tank 2 at 4 gallons per minute. The mixture in tank 2 is pumped back into tank 1 at 1 gallon per minute, and also drains out at 3 gallons per minute. Find the amount of salt in each tank after t minutes.

Let x be the number of pounds of salt dissolved in the first tank at time t and let y be the number of pounds of salt dissolved in the second tank at time t. The rate equations are

$$\frac{dx}{dt} = \left(3\ \frac{\text{gal}}{\text{min}}\right)\left(0\ \frac{\text{lbs}}{\text{gal}}\right) + \left(1\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{y\text{lbs}}{50\text{gal}}\right) - \left(4\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{x\text{lbs}}{50\text{gal}}\right),$$

$$\frac{dy}{dt} = \left(4\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{x\text{lbs}}{50\text{gal}}\right) - \left(1\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{y\text{lbs}}{50\text{gal}}\right) - \left(3\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{y\text{lbs}}{50\text{gal}}\right).$$

Simplify:

$$x' = -0.08x + 0.02y, \quad y' = 0.08x - 0.08y.$$

Next, find the characteristic polynomial:

$$\begin{vmatrix} -0.08 - \lambda & 0.02 \\ 0.08 & -0.08 - \lambda \end{vmatrix} = \lambda^2 + 0.16\lambda + 0.048 = (\lambda + 0.04)(\lambda + 0.12).$$

The eigenvalues are $\lambda = -0.04$, $\lambda = -0.12$.
Consider $\lambda = -0.04$:

$$A + 0.04I = \begin{bmatrix} -0.04 & 0.02 \\ 0.08 & -0.04 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

This says $a - \frac{1}{2}b = 0$, so $a = \frac{1}{2}b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

Set $b = 2$. The eigenvector is $\langle 1, 2 \rangle$.
Now consider $\lambda = -0.12$:

$$A + 0.12I = \begin{bmatrix} 0.04 & 0.02 \\ 0.08 & 0.04 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

This says $a + \frac{1}{2}b = 0$, so $a = -\frac{1}{2}b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}b \\ b \end{bmatrix} = b \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Set $b = 2$. The eigenvector is $\langle -1, 2 \rangle$.
The solution is

$$\vec{x} = c_1 e^{-0.04t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-0.12t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

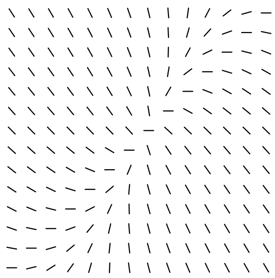
When $t = 0$, $x = 25$ and $y = 0$. Plug in:

$$\begin{bmatrix} 25 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Solving for the constants, I obtain $c_1 = 12.5$, $c_2 = 12.5$. Thus,

$$\vec{x} = 12.5e^{-0.04t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 12.5e^{-0.12t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12.5e^{-0.04t} + 12.5e^{-0.12t} \\ 25e^{-0.04t} - 25e^{-0.12t} \end{bmatrix}.$$

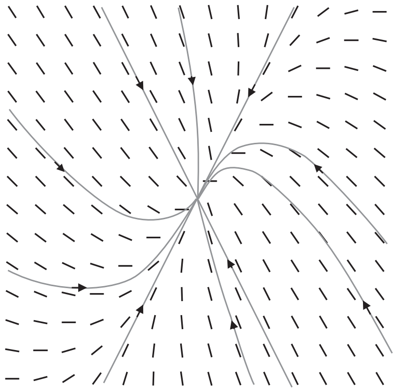
The direction field for the system is shown below:



The picture with the solution curves sketched in is called the **phase portrait**.

The arrows indicate the directions of increasing t. To decide which way the arrows go, take a test point on the positive x-axis — say $x = 5$. Then $x' = -0.08x + 0.02y = -0.4 < 0$, which means x *decreases*. This means the curves' arrows should point to the *left* as the curves cross the x-axis.

Once you get the arrows on part of a curve, you can extend them to the rest of the curves, and the other curves, by consistency.



The eigenvectors have slope 2 ($\langle 1, 2\rangle$) and -2 ($\langle -1, 2\rangle$). These appear as lines (linear solutions) in the phase portrait. □