The exponential of a matrix.

The solution to the exponential growth equation

$$\frac{dx}{dt} = kx \quad \text{is given by} \quad x = c_0 e^{kt}.$$

It is natural to ask whether you can solve a constant coefficient linear system

$$\vec{x}\,' = A\vec{x}$$

in a similar way.

If a solution to the system is to have the same form as the growth equation solution, it should look like

$$\vec{x} = e^{At}\vec{x}_0.$$

The first thing I need to do is to make sense of the **matrix exponential** e^{At}.

The Taylor series for e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

It converges absolutely for all z.

It A is an $n \times n$ matrix with real entries, define

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

The powers A^n make sense, since A is a square matrix. It is possible to show that this series converges for all t and every matrix A.

Differentiating the series term-by-term,

$$\frac{d}{dt} e^{At} = \sum_{n=0}^{\infty} n \frac{t^{n-1} A^n}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{(n-1)!} = A \sum_{n=1}^{\infty} \frac{t^{n-1} A^{n-1}}{(n-1)!} = A \sum_{m=0}^{\infty} \frac{t^m A^m}{m!} = Ae^{At}.$$

This shows that e^{At} solves the differential equation $\vec{x}\,' = A\vec{x}$. The initial condition vector $\vec{x}(0) = \vec{x}_0$ yields the particular solution

$$\vec{x} = e^{At}\vec{x}_0.$$

This works, because $e^{0 \cdot A} = I$ (by setting $t = 0$ in the power series).

Another familiar property of ordinary exponentials holds for the matrix exponential: If A and B commute (that is, $AB = BA$), then

$$e^A e^B = e^{A+B}.$$

You can prove this by multiplying the power series for the exponentials on the left. (e^A is just e^{At} with $t = 1$.)

Example. Compute e^{At} if

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compute the successive powers of A:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \quad \ldots, A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}.$$

Therefore,

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \dfrac{(2t)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \dfrac{(3t)^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}. \square$$

You can compute the exponential of an arbitrary diagonal matrix in the same way:

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}. \square$$

Example. Compute e^{At} if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Compute the successive powers of A:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}, \quad \ldots, A^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$e^{A} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \dfrac{t^n}{n!} & \sum_{n=0}^{\infty} \dfrac{2nt^n}{n!} \\ 0 & \sum_{n=0}^{\infty} \dfrac{t^n}{n!} \end{bmatrix} = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}.$$

Here's where the last equality came from:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t,$$

$$\sum_{n=0}^{\infty} \frac{2nt^n}{n!} = 2t \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = 2t \sum_{m=0}^{\infty} \frac{t^m}{m!} = 2te^t. \square$$

Example. Compute e^{At}, if

$$A = \begin{bmatrix} 3 & -10 \\ 1 & -4 \end{bmatrix}.$$

If you compute powers of A as in the last two examples, there is no evident pattern. Therefore, it would be difficult to compute the exponential using the power series.

Instead, set up the system whose coefficient matrix is A:

$$x' = 3x - 10y,$$

$$y' = x - 4y.$$

The solution is

$$x = c_1 e^t + c_2 e^{-2t}, \quad y = \frac{1}{5} c_1 e^t + \frac{1}{2} c_2 e^{-2t}.$$

Next, note that if B is a 2×2 matrix,

$$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{first column of } B \quad \text{and} \quad B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{second column of } B.$$

In particular, this is true for e^{At}. Now

$$\vec{x} = e^{At}\vec{x}_0$$

is the solution satisfying $\vec{x}(0) = \vec{x}_0$, but

$$\vec{x} = \begin{bmatrix} c_1 e^t + c_2 e^{-2t} \\ \dfrac{1}{5} c_1 e^t + \dfrac{1}{2} c_2 e^{-2t} \end{bmatrix}.$$

Set $\vec{x}(0) = \langle 1, 0 \rangle$ to get the first column of e^{At}:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ \dfrac{1}{5} c_1 + \dfrac{1}{2} c_2 \end{bmatrix}.$$

Hence, $c_1 = \dfrac{5}{3}$, $c_2 = -\dfrac{2}{3}$. So

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{5}{3} e^t - \dfrac{2}{3} e^{-2t} \\ \dfrac{1}{3} e^t - \dfrac{1}{3} e^{-2t} \end{bmatrix}.$$

Set $\vec{x}(0) = \langle 0, 1 \rangle$ to get the second column of e^{At}:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ \dfrac{1}{5} c_1 + \dfrac{1}{2} c_2 \end{bmatrix}.$$

Therefore, $c_1 = -\dfrac{10}{3}$, $c_2 = \dfrac{10}{3}$. Hence,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\dfrac{10}{3} e^t + \dfrac{10}{3} e^{-2t} \\ -\dfrac{2}{3} e^t + \dfrac{5}{3} e^{-2t} \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} \dfrac{5}{3} e^t - \dfrac{2}{3} e^{-2t} & -\dfrac{10}{3} e^t + \dfrac{10}{3} e^{-2t} \\ \dfrac{1}{3} e^t - \dfrac{1}{3} e^{-2t} & -\dfrac{2}{3} e^t + \dfrac{5}{3} e^{-2t} \end{bmatrix}.$$

This method is fine for computing the exponential of a matrix, but notice that you have to solve a system of differential equations to do it. Umm ...isn't that why I wanted e^{At} in the first place? ... $\square$

In some cases, it's possible to use linear algebra to compute the exponential of a matrix. An $n \times n$ matrix A is **diagonalizable** if it has n independent eigenvectors. (This is true, for example, if A has n distinct eigenvalues.)

Suppose A is diagonalizable with independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Let S be the matrix whose columns are the eigenvectors:

$$S = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Then

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D.$$

As I observed above,

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}.$$

On the other hand,

$$e^{Dt} = \sum_{n=0}^{\infty} \frac{t^n (S^{-1}AS)^n}{n!} = S^{-1} \left(\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \right) S = S^{-1} e^{At} S.$$

Hence,

$$e^{At} = S \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} S^{-1}.$$

Example. Compute e^{At} if

$$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}.$$

The eigenvalues are $\lambda =$, $\lambda = -2$. The corresponding eigenvectors are $\langle 5, 1 \rangle$ and $\langle -1, 1 \rangle$. Thus,

$$S = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}.$$

Hence,

$$e^{At} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \left(\frac{1}{6} \right) \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5e^{4t} + e^{-2t} & 5e^{4t} - 5e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + 5e^{-2t} \end{bmatrix}. \quad \square$$

Example. Compute e^{At} if

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}.$$

The eigenvalues are $\lambda = 1$ and $\lambda = 2$ (double); the corresponding eigenvectors are $\langle 3, -1, 3 \rangle$, $\langle 2, 1, 0 \rangle$, $\langle 2, 0, 1 \rangle$.

Therefore,

$$S = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}.$$

From this, it follows that

$$e^{At} = \begin{bmatrix} -3e^{t} + 4e^{2t} & 6e^{t} - 6e^{2t} & 6e^{t} - 6e^{2t} \\ e^{t} - e^{2t} & -2e^{t} + 3e^{2t} & -2e^{t} + 2e^{2t} \\ -3e^{t} + 3e^{2t} & 6e^{t} - 6e^{2t} & 6e^{t} - 5e^{2t} \end{bmatrix}.$$

Here's a quick check on the computation: If you set $t = 0$, you get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

— as you should, since $e^0 = I$.

Note that this check isn't foolproof — just because you get I by setting $t = 0$ doesn't mean your answer is right. However, if you *don't* get I, your answer is surely wrong! $\square$

Next, I'll describe an iterative algorithm for computing e^{At} that only requires that one know the eigenvalues of A.

Let A be an $n \times n$ matrix. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of the eigenvalues, with multiple eigenvalues repeated according to their multiplicity.

Let

$$a_1 = e^{\lambda_1 t},$$

$$a_k = e^{\lambda_k t} \star a_{k-1}(t) = \int_0^t e^{\lambda_k (t-u)} a_{k-1}(u)\, du, \quad k = 2, \ldots, n,$$

$$B_1 = I,$$

$$B_k = (A - \lambda_{k-1} I) \cdot B_{k-1}, \quad k = 2, \ldots, n,$$

Then

$$e^{At} = a_1 B_1 + a_2 B_2 + \ldots + a_n B_n.$$

To prove this, I'll show that the expression on the right satisfies the differential equation $\vec{x}\,' = A\vec{x}$. To do this, I'll need two facts about the characteristic polynomial $p(x)$.

1. $(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) = \pm p(x)$.
2. (**Cayley-Hamilton theorem**) $p(A) = 0$.

Example. I'll illustrate the Cayley-Hamilton theorem with the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

The characteristic polynomial is $(2 - \lambda)(1 - \lambda) - 6 = \lambda^2 - 3\lambda - 4$. The Cayley-Hamilton theorem asserts that if you plug A into $\lambda^2 - 3\lambda - 4$, you'll get the zero matrix.

First,

$$A^2 = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 6 & 7 \end{bmatrix}.$$

Therefore,

$$A^2 - A - 4I = \begin{bmatrix} 10 & 9 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \square$$

Proof of the algorithm. First,

$$a_k = \int_0^t e^{\lambda_k (t-u)} a_{k-1}(u)\, du = e^{\lambda_k t} \int_0^t e^{-\lambda_k u} a_{k-1}(u)\, du.$$

Hence,

$$a_k' = \lambda_k e^{\lambda_k t} \int_0^t e^{-\lambda_k u} a_{k-1}(u)\,du + e^{\lambda_k t} e^{-\lambda_k t} a_{k-1}(t),$$

$$a_k' = \lambda_k a_k + a_{k-1}.$$

Therefore,

$$\begin{aligned}(a_1B_1 + a_2B_2 + \ldots + a_nB_n)' = \\ &\lambda_1 a_1 B_1 + \\ &\lambda_2 a_2 B_2 + a_1 B_2 + \\ &\lambda_3 a_3 B_3 + a_2 B_3 + \\ &\vdots \\ &\lambda_n a_n B_n + a_{n-1} B_n.\end{aligned}$$

Expand the $a_{i-1}B_i$ terms using

$$a_{i-1}B_i = a_{i-1}(A - \lambda_{i-1}I)B_{i-1} = a_{i-1}AB_{i-1} - \lambda_{i-1}a_{i-1}B_{i-1}.$$

Then

$$\begin{aligned}&\lambda_1 a_1 B_1 + \\ &\lambda_2 a_2 B_2 + a_1 A B_1 - \lambda_1 a_1 B_1 + \\ &\lambda_3 a_3 B_3 + a_2 A B_2 - \lambda_2 a_2 B_2 + \\ &\vdots \\ &\lambda_n a_n B_n + a_{n-1} A B_{n-1} - \lambda_{n-1} a_{n-1} B_{n-1} = \\ &\lambda_n a_n B_n + A(a_1 B_1 + a_2 B_2 + \ldots + a_{n-1} B_{n-1}) = \\ \lambda_n a_n B_n - A a_n B_n &+ A(a_1 B_1 + a_2 B_2 + \ldots + a_n B_n) = \\ -a_n(A - \lambda_n I)B_n &+ A(a_1 B_1 + a_2 B_2 + \ldots + a_n B_n) = \\ -a_n p(A) &+ A(a_1 B_1 + a_2 B_2 + \ldots + a_n B_n) = \\ &A(a_1 B_1 + a_2 B_2 + \ldots + a_n B_n).\end{aligned}$$

This shows that $M = a_1B_1 + a_2B_2 + \ldots + a_nB_n$ satisfies $M' = AM$.
Now

$$(e^{-tA}M)' = -Ae^{-tA}M + e^{-tA}AM = -e^{-tA}AM + e^{-tA}AM = 0,$$

since (by the power series) $e^{-tA}A = Ae^{-tA}$. (Remember that matrix multiplication is not commutative in general!) It follows that $e^{-tA}M$ is a constant matrix.

Set $t = 0$. Since $a_2 = \cdots = a_n = 0$, it follows that $M(0) = I$. In addition, $e^{-0\cdot A} = I$. Therefore, $e^{-tA}M = I$, and hence $M = e^{At}$. □

Example. Solve

$$\vec{x}\,' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}, \qquad \vec{x}_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

The characteristic polynomial is $(\lambda - 2)^2$. You can check that there is only one independent eigenvector.
List the eigenvalues: $\{2, 2\}$. Since $\lambda = 2$ is a double root, it is listed twice.
First, I'll compute the a_k's:

$$a_1 = e^{2t},$$

$$a_2 = e^{2t} \star a_1(t) = \int_0^t e^{2(t-u)} e^{2u}\,du = e^{2t} \int_0^t du = te^{2t}.$$

Here are the B_k's:

$$B_1 = I, \quad B_2 = (A - 2I)B_1 = A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Therefore,

$$e^{At} = e^{2t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} e^{2t} + te^{2t} & -te^{2t} \\ te^{2t} & e^{2t} - te^{2t} \end{bmatrix}.$$

As a check, note that setting $t = 0$ produces the identity.)
The solution is

$$\vec{x} = \begin{bmatrix} e^{2t} + te^{2t} & -te^{2t} \\ te^{2t} & e^{2t} - te^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

You get the general solution by replacing $\langle 3, 4 \rangle$ with $\langle c_1, c_2 \rangle$. □

Example. Find e^{At} if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

The eigenvalues are obviously $\lambda = 1$ (double) and $\lambda = 2$.
First, I'll compute the a_k's. $a_1 = e^t$, and

$$a_2 = \int_0^t e^{t-u} e^u \, du = e^t \int_0^t du = te^t,$$

$$a_3 = \int_0^t e^{2(t-u)} ue^u \, du = -te^t - e^t + e^{2t}.$$

Next, I'll compute the B_k's. $B_1 = I$, and

$$B_2 = A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

$$B_3 = (A - I)B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ te^t + 2e^t - 2e^{2t} & e^t - e^{2t} & e^{2t} \end{bmatrix}. \ \square$$

Example. Use the matrix exponential to solve

$$\vec{x}\,' = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix} \vec{x}.$$

The characteristic polynomial is $\lambda^2 + 2\lambda + 2$. The eigenvalues are $\lambda = -1 \pm i$.
First, I'll compute the a_k's. $a_1 = e^{(-1+i)t}$, and

$$a_2 = \int_0^t e^{(-1+i)(t-u)} e^{(-1-i)u} \, du = e^{(-1+i)t} \int_0^t e^{(1-i)u} e^{(-1-i)u} \, du =$$

$$e^{(-1+i)t}\int_0^t e^{-2iu}\,du = e^{(-1+i)t}\frac{i}{2}\left(e^{-2it}-1\right) = \frac{i}{2}\left(e^{(-1-i)t}-e^{(-1+i)t}\right).$$

Next, I'll compute the B_k's. $B_1 = I$, and

$$B_2 = A-(-1+i)I = \begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix}.$$

Therefore,

$$e^{At} = e^{(-1+i)t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i}{2}\left(e^{(-1-i)t}-e^{(-1+i)t}\right)\begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix} = e^{-t}\begin{bmatrix} \cos t+3\sin t & -5\sin t \\ 2\sin t & \cos t-3\sin t \end{bmatrix}.$$

Notice that all the i's have dropped out. The exponential of a real matrix is a real matrix.

Finally, the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{-t}\begin{bmatrix} \cos t+3\sin t & -5\sin t \\ 2\sin t & \cos t-3\sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.\ \square$$

Example. I'll compare the matrix exponential and the eigenvector solution methods by solving the following system both ways:

$$\vec{x}\,' = [\,2 \quad -11 \quad 2\,]\,\vec{x}.$$

The characteristic polynomial is $\lambda^2-4\lambda+5$. The eigenvalues are $\lambda = 2\pm i$.

Consider $\lambda = 2+i$:

$$A-(2+i)I = [\,-i \quad -11 \quad -i\,].$$

As usual, I ignore the second row. I want a vector $\langle a, b\rangle$ such that $(-i)a+(-1)b = 0$. Switch the $-i$ and -1 and negate one of them: $a = 1$, $b = -i$. Thus, $\langle 1, -i\rangle$ is an eigenvector.

The corresponding solution is

$$e^{(2+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}\begin{bmatrix} \cos t+i\sin t \\ \sin t-i\cos t \end{bmatrix}.$$

Take the real and imaginary parts:

$$\operatorname{re} e^{(2+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix},$$

$$\operatorname{im} e^{(2+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}.$$

The solution is

$$\vec{x} = e^{2t}\left(c_1\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}\right).$$

Now I'll solve the equation using the exponential. The eigenvalues are $\{2+i, 2-i\}$. Compute the a_k's. $a_1 = e^{(2+i)t}$, and

$$a_2 = e^{(2-i)t}\star e^{(2+i)t} = \int_0^t e^{(2-i)(t-u)}e^{(2+i)u}\,du = e^{(2-i)t}\int_0^t e^{2iu}\,du =$$

$$e^{(2-i)t}\left[-\frac{i}{2}e^{2iu}\right]_0^t = \frac{i}{2}e^{(2-i)t}\left(1-e^{2it}\right) = \frac{i}{2}e^{2t}\left(e^{-it}-e^{-it}\right) = e^{2t}\sin t.$$

Next, compute the B_k's. $B_1 = I$, and

$$B_2 = A - (2+i)I = [\,-i \quad -11 \quad -i\,].$$

Therefore,

$$e^{At} = e^{(2+i)t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{2t}\sin t\,[\,-i \quad -11 \quad -i\,] = e^{2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

The solution is

$$\vec{x} = e^{2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

In this case, I think the eigenvector approach is easier. □

Example. Solve the system

$$\vec{x}\,' = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\vec{x}.$$

I'll do this using eigenvectors, then using the matrix exponential.

The characteristic polynomial is $\lambda^2 - 2\lambda + 1$. The eigenvalue is $\lambda = 1$ (double).

$$A - I = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix}.$$

Ignore the first row, and divide the second row by 2, obtaining the vector $\langle 1, -2\rangle$. I want $\langle a, b\rangle$ such that $(1)a + (-2)b = 0$. Swap 1 and -2 and negate the -2: I get $\langle a, b\rangle = \langle 2, 1\rangle$. This is an eigenvector for $\lambda = -1$.

Since I only have one eigenvector, I need a generalized eigenvector. This means I need $\langle a', b'\rangle$ such that

$$\begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 4 & -8 & 2 \\ 2 & -4 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This means that $a' = 2b' + \dfrac{1}{2}$. Setting $b' = 0$ yields $a' = \dfrac{1}{2}$. The generalized eigenvector is $\left\langle \dfrac{1}{2}, 0\right\rangle$.

The solution is

$$\vec{x} = c_1 e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\left(te^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right).$$

Next, I'll solve the system using the matrix exponential. The eigenvalues are $\{1, 1\}$. First, I'll compute the a_k's. $a_1 = e^t$, and

$$a_2 = e^t \star e^t = \int_0^t e^{t-u}e^u\,du = e^t\int_0^t du = te^t.$$

Next, compute the B_k's. $B_1 = I$, and

$$B_2 = A - I = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix}.$$

Therefore,

$$e^{At} = e^t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t\begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} e^t + 4te^t & -8te^t \\ 2te^t & e^t - 4te^t \end{bmatrix}.$$

The solution is

$$\vec{x} = \begin{bmatrix} e^t + 4te^t & -8te^t \\ 2te^t & e^t - 4te^t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

In this case, the matrix exponential may be a little bit easier. □