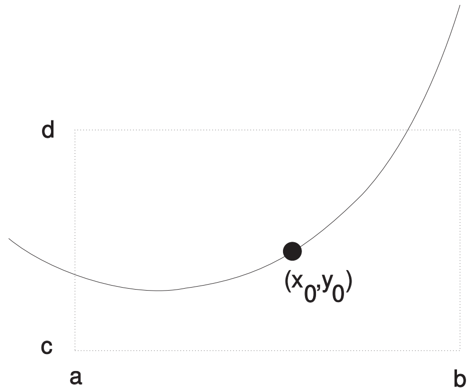
The existence and uniqueness theorem for first-order equations.

Consider a first-order equation

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

Suppose (x_0, y_0) is contained in a rectangle $(a, b) \times (c, d)$ where $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous.

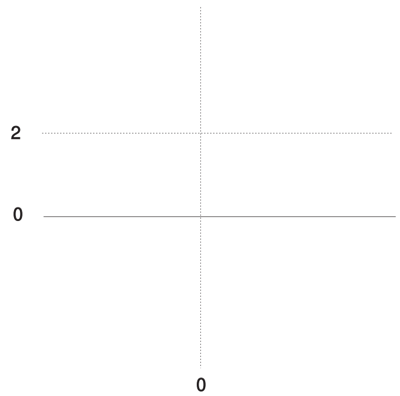


Then there is an interval $(x_0 - h, x_0 + h) \subset (a, b)$ on which there is a unique solution satisfying the initial condition.

Examples. 1. $y' = \dfrac{y^{1/3}}{x(y-2)}$

$$\frac{\partial f}{\partial y} = \frac{-2 - 2y}{3x(y-2)^2 y^{2/3}}.$$

f is continuous for $x \neq 0$ and $y \neq 2$; $\dfrac{\partial f}{\partial y}$ is continuous for $x \neq 0$ and $y \neq 0, 2$. Thus, f and $\dfrac{\partial f}{\partial y}$ are continuous for $x \neq 0$ and $y \neq 0, 2$.



For example, there is a unique solution to the equation satisfying the initial condition $y(3) = 4$. There is no guarantee that there is a unique solution — or any solution — satisfying $y(0) = 7$, $y(1) = 2$, or $y(0) = 0$. $\square$

2. $y' = \dfrac{1}{\sqrt{y - \sin x}}$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}\frac{1}{(y - \sin x)^{3/2}}.$$

f and $\dfrac{\partial f}{\partial y}$ are continuous for $y > \sin x$. Therefore, there is a unique solution passing through every point which lies above the graph of $y = \sin x$. $\square$

3. $\dfrac{dy}{dx} = y^2 + 1$, $y(0) = 1$.

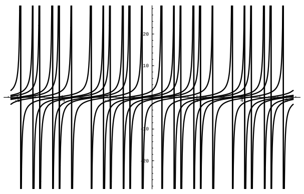
Separate variables:

$$\int \frac{dy}{y^2 + 1} = \int dx.$$

Integrate and solve:

$$\arctan y = x + C, \quad y = \tan(x + C).$$

Here is the family of solution curves:



Plug in the initial condition:

$$1 = y(0) = \tan C.$$

There are infinitely many values of C which satisfy this equation. For example, $\tan \dfrac{\pi}{4} = 1$, but $\tan \dfrac{5\pi}{4} = 1$, too.

However, the solution must be defined on an interval containing the initial condition. The function $\tan(x + C)$ is defined on intervals of the form $C - \dfrac{\pi}{2} < x < C + \dfrac{\pi}{2}$.

$C = \dfrac{\pi}{4}$ yields the interval $-\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$, which contains $x = 0$. (On the other hand, $C = \dfrac{5\pi}{4}$ yields the interval $\dfrac{3\pi}{4} < x < \dfrac{7\pi}{4}$, which doesn't contain $x = 0$.) The solution is

$$y = \tan\left(x + \frac{\pi}{4}\right). \square$$