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Eigenvalues and eigenvectors.

If A is an $n \times n$ matrix, the **characteristic polynomial** of A is

$$p(\lambda) = \det(A - \lambda I).$$

The roots of the characteristic polynomial are called the **eigenvalues** or **characteristic values** of A.

Example. Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 3 \\ 2 & -4 - \lambda \end{bmatrix},$$

$$|A - \lambda I| = (1 - \lambda)(-4 - \lambda) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2).$$

The eigenvalues are $\lambda = -5$ and $\lambda = 2$. $\square$

Example. Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix},$$

$$|A - \lambda I| = \lambda^2 + 4.$$

The eigenvalues are $\lambda = \pm 2$

The eigenvalues are $\lambda = 1$ (double) and $\lambda = 2$. Notice that I *didn't* multiply the $1 - \lambda$ term by the other stuff, since I want the result in factored form.

You may perform both row *and* column operations to simplify a determinant. (You *can't* use column operations to do other things, such as transforming a matrix to row reduced echelon form.)

The idea is to create a row or a column with lots of zeros, then expand by cofactors of that row or column.

Here is how the various row and column operations affect the determinant:

1. Swapping two rows or columns multiplies the determinant by -1.
2. Multiplying a row or a column by a number multiplies the determinant by that number.
3. Adding a multiple of a row or a column to another row or column, respectively, does not change the determinant.

I used row and column operations of the third type in the computation above. □

If λ is an eigenvalue for the $n \times n$ matrix A, an **eigenvector** or **characteristic vector** corresponding to λ is a *nonzero* vector v such that

$$(A - \lambda I)$$

I may take b to be any nonzero number. Take $b = 2$ to kill the fraction; this gives $\langle -1, 2\rangle$ for the eigenvector.

Observe that any nonzero multiple of an eigenvector is an eigenvector — there isn't a unique "answer" here.

Consider $\lambda = 2$. I'll follow the procedure I used for the first eigenvalue.

$$A - 2I = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

Here's a shortcut that's often useful. *This applies to 2×2 matrices only!*

If A is a 2×2 matrix and λ is an eigenvalue of A, then the rows of $A - \lambda I$ must be multiples of each other. Hence, in this case, you may simply kill one of the rows, replacing it with the zero row. (This corresponds to subtracting a multiple of one row from another.)

This corresponds to the equation

$$a - 3b = 0,$$

so $a = 3b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3b \\ b \end{bmatrix} = b \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Take $b = 1$; our eigenvector is $\langle 3, 1\rangle$. $\square$

Example. The situation is the same for larger matrices.

$$A = \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 3 \\ 2 & -2 & -2 \end{bmatrix}$$

I obtained eigenvalues $\lambda = 1$ (double) and $\lambda = 2$ earlier.

First, consider $\lambda = 1$:

$$A - I = \begin{bmatrix} 2 & -2 & -3 \\ -2 & 2 & 3 \\ 2 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Note that the fact that all three rows are multiples of one another is an accident!) This gives

$$a - b - \frac{3}{2}\lambda = 0.$$

Thus, $a = b + \frac{3}{2}\lambda$. Therefore,

$$\begin{bmatrix} a \\ b \\ \lambda \end{bmatrix} = \begin{bmatrix} b + \frac{3}{2}\lambda \\ b \\ \lambda \end{bmatrix} = b \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda \cdot \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Setting $b = 1$, $\lambda = 0$ gives $\langle 1, 1, 0\rangle$. Setting $b = 0$, $\lambda = 1$ gives$\left\langle \frac{3}{2}, 0, 1 \right\rangle$.

Notice that there are two eigenvectors for this eigenvalue, one *independent eigenvector* for each of the free variables. However, it's possible in this situation to only get one eigenvector even thought the eigenvalue is a double root.

Now consider $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 1 & 3 \\ 2 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \mathrm{H} \end{bmatrix}$$

This yields the equations

$$\begin{array}{ccccccc} a & & & - & \lambda & = & 0 \\ & & b & + & \lambda & = & 0 \end{array}$$

Therefore, $a = \lambda$ and $b = -\lambda$. So

$$\begin{bmatrix} a \\ b \\ \lambda \end{bmatrix} = \begin{bmatrix} \lambda \\ -\lambda \\ \lambda \end{bmatrix} = \lambda \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Setting $\lambda = 1$ produces the eigenvector $\langle 1, -1, 1 \rangle$. $\square$