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# Constant coefficient equations: Real and repeated roots.

The full description of these equations is: Linear constant coefficient homogeneous equations. The equations described in the title have the form

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

Here $a_n, \ldots, a_0$ are *constants*. *Linear* means the equation is a sum of the derivatives of $y$, each multiplied by $x$ stuff. (In this case, the $x$ stuff is constant.) *Homogeneous* means that the right side is 0 — there's no term involving only $x$. (Nonhomogeneous equations will be tackled later on.)

Sometimes I'll divide through by the leading coefficient $a_n$ to put the equation in the form

$$y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_2 y'' + b_1 y' + b_0 y = 0.$$

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**Example.** The following equations are linear homogeneous equations with constant coefficients:

$$y''' + 3y'' + 3y' + y = 0$$

$$y'' - 5y' - 6y = 0$$

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The form of the equation makes it reasonable that a solution should be a function whose derivatives are constant multiples of itself. $e^{rx}$ is such a function:

$$\frac{d}{dx} e^{rx} = re^{rx}, \quad \frac{d}{dx^2} e^{rx} = r^2 e^{rx}, \quad \ldots, \quad \frac{d^n}{dx^n} e^{rx} = r^n e^{rx}.$$

Plug $e^{rx}$ into

$$y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_2 y'' + b_1 y' + b_0 y = 0.$$

The result:

$$r^n e^{rx} + b_{n-1} r^{n-1} e^{rx} + \cdots + b_2 r^2 e^{rx} + b_1 r e^{rx} + b_0 e^{rx} = 0.$$

Factor out $e^{rx}$ and cancel it. This leaves

$$r^n + b_{n-1} r^{n-1} + \cdots + b_2 r^2 + b_1 r + b_0 = 0.$$

$e^{rx}$ should be a solution to the original equation when $r$ is a root of this polynomial. It is called the **characteristic polynomial**; as the derivation showed, it's obtained by making a polynomial with the coefficients in the original differential equation.

At this point, it isn't obvious why this procedure gives *all* the solutions to the equation. (How do you know the only possibilities are exponentials?) There are also complications if there are multiple roots or complex roots. For the moment, I'll charge ahead with some examples which illustrate the easy case.

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**Examples.** In these examples, I'll consider the case where the characteristic polynomial has **real, distinct roots**. Each root $r$ gives rise to an $e^{rx}$ term in the general solution. Each $e^{rx}$ is multiplied by an arbitrary constant, and there is one arbitrary constant for each degree of the equation.

1. $y'' - 5y' - 6y = 0$.

The characteristic polynomial is $m^2 - 5m - 6$; solving $m^2 - 5m - 6 = 0$ yields $(m-6)(m+1) = 0$, so $m = 6$ or $m = -1$. The general solution is

$$y = c_1 e^{6x} + c_2 e^{-x}.$$

You can check this by plugging back in. Here are the derivatives:

$$y = c_1 e^{6x} + c_2 e^{-x}, \quad y' = 6c_1 e^{6x} - c_2 e^{-x}, \quad y'' = 36c_1 e^{6x} + c_2 e^{-x}.$$

Therefore,

$$y'' - 5y' + 6y = \left(36c_1 e^{6x} + c_2 e^{-x}\right) - 5\left(6c_1 e^{6x} - c_2 e^{-x}\right) - 6\left(c_1 e^{6x} + c_2 e^{-x}\right) = 0. \square$$

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2. You may sometimes see constant coefficient equations expressed using "$D$" notation. "$D$" means "$\frac{d}{dx}$"; thus,

$$Dy = \frac{dy}{dx} = y', \quad D^2 y = \frac{d^2 y}{dx^2} = y'', \quad D^3 y = \frac{d^3 y}{dx^3} = y''', \ldots.$$

For example, I'll solve $(D^4 - 9D^2 + 20)y = 0$.

It's easy to write down the characteristic equation: just replace the $D$'s with $m$'s:

$$m^4 - 9m^2 + 20 = 0, \quad (x^2-4)(x^2-5) = 0, \quad (x-2)(x+2)(x-\sqrt{5})(x+\sqrt{5}) = 0.$$

The roots are $\pm 2$ and $\pm\sqrt{5}$. (Don't fall into the trap of assuming that roots must be integers, or even rationals!) The solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{\sqrt{5}x} + c_4 e^{-\sqrt{5}x}. \square$$

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What happens if there are repeated roots? Look at the equation $y'' - 2y' + y = 0$. The characteristic equation is $x^2 - 2x + 1 = 0$, which has $x = 1$ as a double root. It is true that $e^x$ is a solution, but it would be incorrect to write $y = c_1 e^x + c_2 e^x$.

The terms $c_1 e^x$ and $c_2 e^x$ are redundant — you could combine them to get $y = (c_1 + c_2)e^x = c_3 e^x$.

It is reasonable to suppose that for a second order equation you should have two *different* solutions. $e^x$ is one; how can you find another?

The idea is to guess the form that such a solution might take. Guess:

$$y = f(x)e^x,$$

i.e. something times the known solution $e^x$. What should $f$ be?

To find $f$, plug $f(x)e^x$ into the equation. Here are the derivatives:

$$y = f(x)e^x, \quad y' = f'(x)e^x + f(x)e^x, \quad y'' = f''(x)e^x + 2f'(x)e^x + f(x)e^x.$$

Plug them in:

$$y'' - 2y' + y = (f''(x)e^x + 2f'(x)e^x + f(x)e^x) - 2(f'(x)e^x + f(x)e^x) + f(x)e^x = f''(x)e^x = 0.$$

Hence, $f''(x) = 0$. Integrate twice and obtain $f(x) = c_1 + c_2 x$. Thus,

$$y = c_1 e^x + c_2 x e^x.$$

In fact, this is the general solution — notice the two arbitrary constants. The functions $e^x$ and $xe^x$ are indpendent solutions to the original equation.

In general, if $r$ is a repeated root of multiplicity $k$ in the characteristic polynomial, you get terms $e^{rx}$, $xe^{rx}$, ..., $x^{k-1}e^{rx}$ in the general solution.

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**Example.** $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$

The characteristic equation is $m^3 - 3m^2 + 3m - 1 = 0$, which has $m = 1$ as a root with multiplicity 3. The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x. \ \Box$$

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**Example.** $(D^2 - 1)(D^2 - D - 2)y = 0.$

The characteristic equation is $(m^2-1)(m^2-m-2) = 0$, or $(m-1)(m+1)^2(m-2) = 0$. The roots are 1, 2, and $-1$ (double). The general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{-x} + c_4 x e^{-x}.$$

Note: You can write the terms in the solution in any order you please. Nor does it matter which "$c$" goes with which term, since they *are* arbitrary constants. $\Box$

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**Example.** $y'' - 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.

The characteristic equation is $x^2 - 4x + 4 = 0$, or $(x-2)^2 = 0$. The root is $x = 2$ (double), and the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x}.$$

To plug in the initial conditions, I need the derivative: $y' = (2c_1 + c_2)e^{2x} + 2c_2 x e^{2x}$. Then

$$2 = y(0) = c_1, \quad 1 = y'(0) = 2c_1 + c_2.$$

These equations yield $c_1 = 2$ and $c_2 = -3$, so

$$y = 2e^{2x} - 3xe^{2x}. \ \Box$$

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**Example.** (Linear systems) Consider the system of differential equations

$$\frac{dx}{dt} = x' = x + 4y, \quad \frac{dy}{dt} = y' = 2x + 3y.$$

I want to solve for $x$ and $y$ in terms of $t$.

Solve the second equation for $x$:

$$x = \frac{1}{2}\left(y' - 3y\right).$$

Differentiate:

$$x' = \frac{1}{2}\left(y'' - 3y'\right).$$

Plug the expressions for $x$ and $x'$ into the first equation:

$$\frac{1}{2}(y'' - 3y') = \frac{1}{2}(y' - 3y) + 4y.$$

Simplify:

$$y'' - 4y' - 5y = 0.$$

The characteristic equation is $m^2 - 4m - 5 = 0$, or $(m - 5)(m + 1) = 0$. The roots are $m = 5$ and $m = -1$. Therefore,

$$y = c_1 e^{5t} + c_2 e^{-t}.$$

Now $y' = 5c_1 e^{5t} - c_2 e^{-t}$, so

$$x = \frac{1}{2}(y' - 3y) = \frac{1}{2}\left[\left(5c_1 e^{5t} - c_2 e^{-t}\right) - 3\left(c_1 e^{5t} + c_2 e^{-t}\right)\right] = c_1 e^{5t} - 2c_2 e^{-t}.$$

There are other ways of solving linear systems, but for small systems brute force works reasonably well! $\square$

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## Constant coefficient equations: Complex roots.

A linear homogeneous equations with constant coefficients is an equation of the form

$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0.$$

Here $a_n$, ..., $a_0$ are constants. You can obtain the general solution by examining the **characteristic equation**

$$a_n m^n + \cdots a_1 m + a_0 = 0.$$

I've already described what to do if the equation has distinct real roots or multiple real roots.

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**Example.** $y''' - 5y'' + 8y' - 4y = 0$.

The characteristic equation is $x^3 - 5x^2 + 8x - 4 = 0$, or $(x - 1)(x - 2)^2 = 0$. The roots are $x = 1$ and $x = 2$ (double). The general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}. \square$$

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Now suppose the characteristic equation has a complex root $a + bi$. From basic algebra, complex roots come in **conjugate pairs**: $a + bi$ and $a - bi$. It's reasonable to expect solutions

$$c_1 e^{(a+bi)x} \quad \text{and} \quad c_2 e^{(a-bi)x}.$$

However, these are complex solutions, and you should have real solutions to the original real differential equation. I'll use the **complex exponential formula**

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbf{R}.$$

You can derive this formula by considering the Taylor series for $e^{i\theta}$, $\cos\theta$, and $\sin\theta$.

Now

$$c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x} = c_1 e^{ax} e^{ibx} + c_2 e^{ax} e^{-ibx} =$$

$$e^{ax}\left(c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)\right) = e^{ax}\left((c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx\right).$$

Let $c_3 = c_1 + c_2$ and $c_4 = i(c_1 - c_2)$. Observe that $c_1$ and $c_2$ can be solved for in terms of $c_3$ and $c_4$, so no generality is lost with this substitution. Then

$$c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x} = e^{ax}(c_3 \cos bx + c_4 \sin bx).$$

Each pair of conjugate complex roots $a \pm bi$ in the characteristic equation generates a pair of independent solutions of this form.

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**Example.** $y'' + y = 0$.

The characteristic equation $m^2 + 1 = 0$ has roots $\pm i$. The solution is

$$y = c_1 \cos x + c_2 \sin x. \ \square$$

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**Example.** $y'' + 2y' + 5y = 0$.

The characteristic equation $m^2 + 2m + 5 = 0$ has roots $m = -1 \pm 2i$. The solution is

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x). \ \square$$

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**Example.** $y''' + y = 0$.

The characteristic polynomial $m^3 + 1$ factors into $(m+1)(m^2 - m + 1)$. The roots are $m = -1$ and $m = \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$. The solution is

$$y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right). \ \square$$

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**Example.** $(D^2 + 4)^2 y = 0$.

The characteristic equation $(m^2 + 4)^2 = 0$ has repeated complex roots: $m = \pm 2i$ (each double). The solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x.$$

(What's the solution to $(D^2 + 2D + 5)^2 y = 0$?) $\square$