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# The wave equation.

An elastic string is stretched between two supports. The supports are at $x = 0$ and $x = L$. The string is set in motion (for example, by plucking it). The vertical displacement of the piece of the string at position $x$ at time $t$ is $u(x, t)$.

The displacement satisfies the **wave equation**

$$u_{tt} = a^2 u_{xx}.$$

Here $a^2 = \dfrac{T^2}{\rho}$, where $T$ is the tension in the string and $\rho$ is the string's density.

The general problem is to solve for $u$, given various initial conditions and boundary conditions.

Consider the case where the ends of the string are fixed. This gives the boundary conditions

$$u(0, t) = u(L, t) = 0, \qquad t > 0.$$

The initial conditions will normally be given in the form of the string's initial displacement and velocity:

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \qquad 0 < x < L.$$

In order for these conditions to be consistent with the boundary conditions,

$$f(0) = f(L) = g(0) = g(L) = 0.$$

Use separation of variables. Let $u(x, t) = p(x)q(t)$. Then

$$p(x)q''(t) = a^2 p''(x)q(t), \quad \frac{q''(t)}{q(t)} = a^2 \frac{p''(x)}{p(x)}.$$

The left side depends on $t$ while the right side depends on $x$. Therefore, both sides are constant:

$$\frac{q''(t)}{q(t)} = a^2 \frac{p''(x)}{p(x)} = -\lambda^2.$$

Therefore,

$$q''(t) + \lambda^2 q(t) = 0, \quad p''(x) + \frac{\lambda^2}{a^2} p(x) = 0.$$

At this point, $\lambda^2$ could be positive, negative, or 0. It's possible to show that if $\lambda^2 \le 0$, then $p \equiv 0$, whence $u \equiv 0$. It's reasonable to assume that the solution is nontrivial, so these cases may be ruled out. Therefore, $\lambda^2 > 0$, and

$$p(x) = c_1 \cos\left(\frac{\lambda x}{a}\right) + c_2 \sin\left(\frac{\lambda x}{a}\right).$$

The boundary conditions give

$$0 = u(0, t) = p(0)q(t), \quad p(0) = 0, \qquad 0 = u(L, t) = p(L)q(t), \quad p(L) = 0.$$

(If $q(t) \equiv 0$, then $u \equiv 0$, which again gives the trivial solution.)

Plug in the boundary conditions: $0 = p(0) = c_1$. Hence,

$$p(x) = c_2 \sin\left(\frac{\lambda x}{a}\right).$$

Likewise,

$$p(x) = c_2 \sin\left(\frac{\lambda x}{a}\right), \quad 0 = p(L) = c_2 \sin\left(\frac{\lambda L}{a}\right).$$

If $c_2 = 0$, then $p(x) \equiv 0$, which gives the trivial solution again. Therefore,

$$\sin\left(\frac{\lambda L}{a}\right) = 0, \quad \frac{\lambda L}{a} = n\pi, \quad n = 1, 2, \ldots.$$

Therefore, $\lambda = \dfrac{\pi n a}{L}$.
Thus,

$$p(x) = c_2 \sin\left(\frac{\pi n x}{L}\right).$$

Now solve for $q$:

$$q''(t) + \lambda^2 q(t) = 0.$$

Since $\lambda^2$ is now known to be positive,

$$q(t) = d_1 \cos\left(\frac{\pi n a t}{L}\right) + d_2 \sin\left(\frac{\pi n a t}{L}\right).$$

Thus, the fundamental solutions to the equation have the form

$$\sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi n a t}{L}\right), \quad \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi n a t}{L}\right).$$

By superposition, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{L}\right)\left(b_n \cos\left(\frac{\pi n a t}{L}\right) + k_n \sin\left(\frac{\pi n a t}{L}\right)\right).$$

Consider the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

The first equation gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right).$$

Thus, $b_n$ is the $n$-th Fourier sine coefficient of $f(x)$:

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{\pi n x}{L}\, dx.$$

To insert the second initial condition, differentiate the series for $u$ term-by-term with respect to $t$. (This is justified, for example, if the series converges absolutely.) Then

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{\pi n a}{L} \sin\left(\frac{\pi n x}{L}\right)\left(-b_n \sin\left(\frac{\pi n a t}{L}\right) + k_n \cos\left(\frac{\pi n a t}{L}\right)\right).$$

Plug in $u_t(x,0) = g(x)$:

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} k_n \frac{\pi n a}{L} \sin\left(\frac{\pi n x}{L}\right).$$

This means that $k_n \dfrac{\pi n a}{L}$ is the $n$-th Fourier sine coefficient for $g(x)$.

I'll consider an important case to illustrate this. Suppose that the string is initially at rest. This means that

$$u_t(x,0) = 0.$$

Then

$$0 = u_t(x,0) = \sum_{n=1}^{\infty} k_n \frac{L}{\pi n a} \sin\left(\frac{\pi n x}{L}\right).$$

The Fourier sine expansion for 0 has all coefficients 0, so $k_n = 0$ for all $n$. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \cos\left(\frac{\pi n a t}{L}\right).$$

Here $b_n$ is the $n$-th Fourier sine coefficient for $u(x,0) = f(x)$.

**Example.** Solve the wave equation

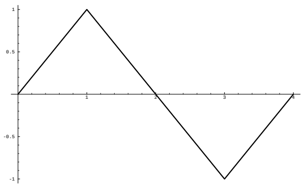
$$u_{tt} = 4u_{xx}, \qquad 0 < x < 4, \quad t > 0$$

$$u(0,t) = u(4,t) = 0, \qquad t > 0$$

$$u(x,0) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 \le x < 3 \\ x - 4 & \text{if } 3 \le x < 4 \end{cases}$$

$$u_t(x,0) = 0, \qquad 0 < x < 4$$

Here's a picture of the initial distribution:



The solution will be

$$u(x,t) = \sum_{n+1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \cos\left(\frac{\pi n a t}{L}\right),$$

where $b_n$ is the $n$-th Fourier sine coefficient of $u(x,0)$.

Find $b_n$:

$$b_n = \frac{2}{4} \int_0^4 f(x) \sin\left(\frac{\pi n x}{4}\right)\, dx =$$

$$\frac{1}{2}\left(\int_0^1 x \sin\left(\frac{\pi n x}{4}\right)\, dx + \int_1^3 (2 - x) \sin\left(\frac{\pi n x}{4}\right)\, dx + \int_3^4 (x - 4) \sin\left(\frac{\pi n x}{4}\right)\, dx\right) =$$

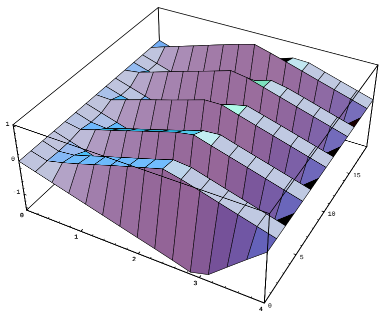
$$\frac{1}{2}\left[-\frac{4}{\pi n}x\cos\left(\frac{\pi nx}{4}\right)+\frac{16}{\pi^2n^2}\sin\left(\frac{\pi nx}{4}\right)\right]_0^1+\frac{1}{2}\left[-\frac{4}{\pi n}(2-x)\cos\left(\frac{\pi nx}{4}\right)-\frac{16}{\pi^2n^2}\sin\left(\frac{\pi nx}{4}\right)\right]_1^3+$$

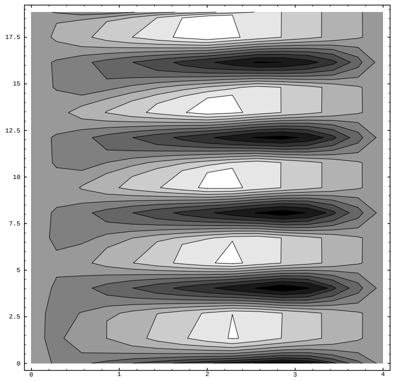
$$\frac{1}{2}\left[-\frac{4}{\pi n}(x-4)\cos\left(\frac{\pi nx}{4}\right)+\frac{16}{\pi^2n^2}\sin\left(\frac{\pi nx}{4}\right)\right]_3^4=\frac{16}{\pi^2n^2}\left(\sin\left(\frac{\pi n}{4}\right)-\sin\left(\frac{3\pi n}{4}\right)\right).$$

Hence, the solution is

$$u(x,t)=\sum_{n=1}^{\infty}\frac{16}{\pi^2n^2}\left(\sin\left(\frac{\pi n}{4}\right)-\sin\left(\frac{3\pi n}{4}\right)\right)\sin\left(\frac{\pi nx}{4}\right)\cos\left(\frac{\pi nt}{2}\right).$$

I'll plot the solution, using the first 20 terms of the sum to approximate $u(x,t)$. The picture on the left shows the graph of $u(x,t)$ for $0\le t\le 6\pi$; since it's a little hard to see what it's doing, I've also drawn the contour plot (on the right).




(The dark areas represent negative heights while the light areas represent positive heights.) You can see the middle of the string oscillating up and down (the alternating light and dark areas as you move from bottom to top across the contour plot). □

**Example.** Solve the wave equation

$$u_{tt}=4u_{xx},\qquad 0<x<4,\quad t>0$$

$$u(0,t)=u(4,t)=0,\qquad t>0$$

$$u(x,0)=\begin{cases}x & \text{if } 0<x<1\\ 2-x & \text{if } 1\le x<3\\ x-4 & \text{if } 3\le x<4\end{cases}$$

$$u_t(x,0)=\sin\frac{\pi x}{2},\qquad 0<x<4$$

This time, the solution is

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{\pi nx}{L}\right)\left(b_n\cos\left(\frac{\pi nat}{L}\right)+k_n\sin\left(\frac{\pi nat}{L}\right)\right).$$

Here $b_n$ is the $n$-th Fourier sine coefficient of $u(x,0)$ and $k_n\dfrac{\pi na}{L}$ is the $n$-th Fourier sine coefficient for $\sin\dfrac{\pi x}{2}$.

I computed the $b_n$ in the first problem:

$$b_n = \frac{16}{\pi^2 n^2}\left(\sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{3\pi n}{4}\right)\right).$$

Next,

$$\frac{2\pi n}{4} k_n = \frac{2}{4}\int_0^4 \sin\frac{\pi x}{2}\sin\frac{\pi n x}{2}\,dx.$$

Suppose first that $n \ne 1$. Use

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right).$$

Therefore,

$$\pi n k_n = \frac{1}{2}\int_0^4 \left(\cos\frac{\pi(1-n)x}{2} - \cos\frac{\pi(1+n)x}{2}\right)\,dx =$$

$$\frac{1}{2}\left[\frac{2}{\pi(1-n)}\sin\frac{\pi(1-n)x}{2} - \frac{2}{\pi(1+n)}\sin\frac{\pi(1+n)x}{2}\right]_0^4 = 0.$$

If $n = 1$,

$$\pi k_1 = \frac{1}{2}\int_0^4 (1 - \cos\pi x)\,dx = 2.$$

The solution is

$$u(x,t) = 2\sin\frac{\pi x}{4}\sin\frac{\pi t}{2} + \sum_{n=2}^{\infty}\frac{16}{\pi^2 n^2}\left(\sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{3\pi n}{4}\right)\right)\sin\left(\frac{\pi n x}{4}\right)\cos\left(\frac{\pi n t}{2}\right). \square$$