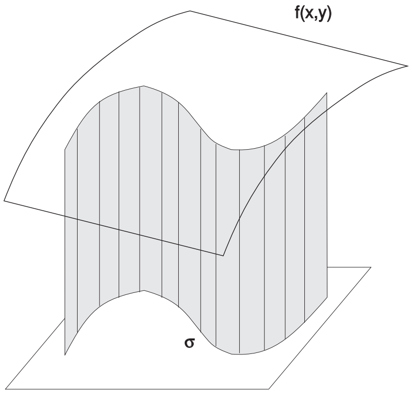
Path integrals.

Here is a picture of a rectangle sum approximation to a **path integral**

$$\int_{\vec{\sigma}} f\, ds.$$

($\vec{\sigma}$ is a curve in the x-y-plane.)



The curve is divided into little pieces. A small piece of the curve has length ds, where $ds = \sqrt{dx^2 + dy^2}$.

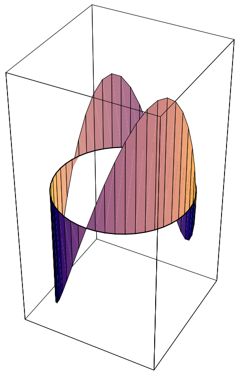
To compute a path integral, represent the curve in parametrized form: $\sigma(t) = (x(t), y(t))$, or $x = x(t)$, $y = y(t)$. Then operating formally with the differentials,

$$\sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Use $x = x(t)$, $y = y(t)$ to write $f(x, y)$ in terms of t. Finally, if $a \le t \le b$ for the path σ, then these inequalities give the limits of integration.

Examples.

1. Compute $\displaystyle\int_{\vec{\sigma}} f\, ds$, where $f(x, y) = x^3 - 3xy$ and $\vec{\sigma}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$.



First, I'll find ds:

$$\vec{\sigma}\,'(t) = \langle -\sin t, \cos t\rangle, \text{ so } |\vec{\sigma}\,'(t)| = \sqrt{(\sin t)^2 + (\cos t)^2} = 1.$$

Therefore, $ds = 1 \cdot dt = dt$.
Next, I convert f to a function of t:

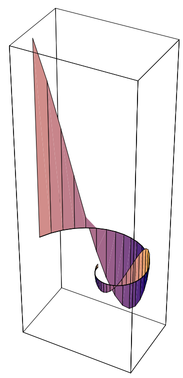
$$f(t) = (\cos t)^3 - 3(\cos t)(\sin t).$$

Therefore,

$$\int_{\vec{\sigma}} f\, ds = \int_0^{2\pi} \left((\cos t)^3 - 3(\cos t)(\sin t)\right)\, dt = 0.$$

Geometrically, this means that the area above the curve equals the signed area below the curve. These areas are signed, so they cancel each other. □

2. Compute $\displaystyle\int_{\vec{\sigma}} f\, ds$, where $f(x, y) = x^2 - y^2$ and $\vec{\sigma}(t) = \langle e^t \cos 3t, e^t \sin 3t\rangle$, $-1 \le t \le 1$.



First,

$$\vec{\sigma}\,'(t) = \langle e^t \cos 3t - 3e^t \sin 3t, e^t \sin 3t + 3e^t \cos 3t\rangle.$$

After some amusing algebra, I get

$$|\vec{\sigma}\,'(t)| = \sqrt{10e^{2t}} = \sqrt{10}e^t.$$

Therefore, $ds = \sqrt{10}e^t\, dt$.
Next,

$$f(t) = e^{2t}(\cos 3t)^2 - e^{2t}(\sin 3t)^2 = e^{2t} \cos 6t.$$

Hence,

$$\int_{\vec{\sigma}} f\, ds = \int_{-1}^{1} e^{2t} \cos 6t \cdot \sqrt{10}e^t\, dt = \frac{\sqrt{10}}{15}\left(e^3 \cos 6 + 2e^3 \sin 6 - e^{-3} \cos 6 + 2e^{-3} \sin 6\right) \approx 1.68349.$$ □

3. A wire is bent into the shape of the helix

$$\vec{\sigma}(t) = \langle \cos t, \sin t, t\rangle, \quad 0 \le t \le 4\pi.$$

The density is proportional to the square of the distance from the origin. Find the mass of the wire.

In this case, the curve is 3-dimensional, so I can't picture it as a "fence" as I did with the 2-dimensional curves. However, the computation is essentially the same.

The density is $\delta = k(x^2 + y^2 + z^2)$, where k is a constant. The mass is just $\displaystyle\int_{\vec{\sigma}} \delta\, ds$.
The velocity is

$$\vec{\sigma}\,'(t) = \langle -\sin t, \cos t, 1\rangle, \text{ so } |\vec{\sigma}\,'(t)| = \sqrt{(\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

Hence, $ds = \sqrt{2}\, dt$.
Write δ in terms of t:

$$\delta = k\left[(\cos t)^2 + (\sin t)^2 + 1\right] = k(1 + t^2).$$

The mass is

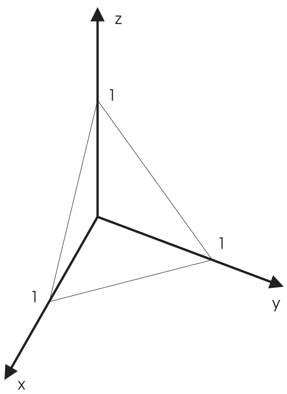
$$\int_0^{4\pi} k(1 + t^2) \cdot \sqrt{2}\, dt = k\sqrt{2}\left(4\pi \frac{64}{3}\pi^3\right). \ \square$$

Problems on Path Integrals.

This is Problem Set 10. It is due on Monday, September 20.

1. Compute $\displaystyle\int_{\vec{\sigma}} f\, ds$, where $\vec{\sigma} = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, and $f(x, y) = |xy|$.

2. A wire is made of the three segments connecting the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.



The density of the wire is $\delta = x + y + z$. Find its mass.

3. Suppose $\displaystyle\int_{\vec{\sigma}} f\, ds = 42$, where $\vec{\sigma} = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Let $\phi(t) = \langle \cos(2\pi - t), \sin(2\pi - t)\rangle$, $0 \le t \le 2\pi$ (so ϕ is the same curve as σ, but traversed in the opposite direction). What is $\displaystyle\int_{\vec{\phi}} f\, ds$? Explain.