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# Complex numbers.

A **complex number** has the form $a + bi$, where $a$ and $b$ are real numbers, and $i$ denotes $\sqrt{-1}$. $a$ is called the **real part** of $a + bi$ and $b$ is called the **imaginary part**. Notation:

$$a = \operatorname{re}(a + bi), \qquad b = \operatorname{im}(a + bi).$$

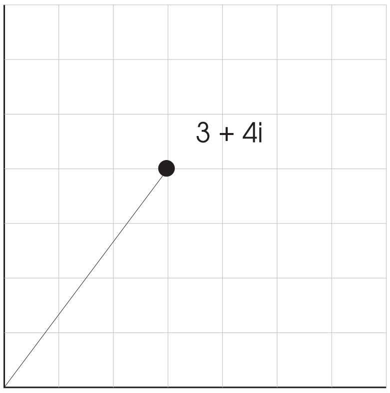
You perform arithmetic on complex numbers in the obvious way. Addition is easy: Just add the real parts and the imaginary parts separately:

$$(17 - 2i) + (3 + \sqrt{2}i) = 20 + (-2 + \sqrt{2})i.$$

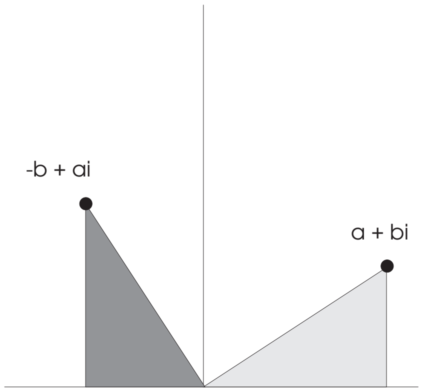
Multiply by using the distributive law:

$$(1 + 2i)(4 - 3i) = 1 \cdot 4 + 1 \cdot (-3i) + 2i \cdot 4 + 2i \cdot (-3i) = 4 - 3i + 8i + 6 = 10 + 5i.$$

Complex numbers are often represented by points on the plane, with the real part in the $x$ direction and the complex part in the $y$ direction. For example, here is the complex number $3 + 4i$:




**Example.** $i \cdot (a + bi) = -b + ai$. Picture:




As you can see, *multiplying a complex number by $i$ rotates it by $\dfrac{\pi}{2}$ in the complex plane.* $\square$

Two complex numbers $a + bi$ and $c + di$ are equal if and only if their real parts are equal and their imaginary parts are equal: $a = c$ and $b = d$. Hence, in order to tell whether two complex numbers are equal, you usually write them in "$a + bi$" form.

**Example.** $\dfrac{1}{i} = \dfrac{1}{i} \cdot \dfrac{i}{i} = \dfrac{i}{i^2} = \dfrac{i}{-1} = -i$. □

**Example.** A quotient of two complex numbers, such as $\dfrac{1+2i}{3-4i}$, can be written in the form $a + bi$ by multiplying top and bottom by the **conjugate** of the denominator. The conjugate of $x + iy$ is $x - iy$, so

$$\frac{1+2i}{3-4i} = \frac{1+2i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{-5+10i}{25} = -\frac{1}{5} + i\frac{2}{5}.$$

The conjugate of the complex number $z$ is often denoted $\overline{z}$. □

**Example.** The **length** of a complex number $a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$. For example,

$$|12 - 5i = \sqrt{144 + 25} = 13.$$

There is a useful relationship between a number, its length, and its conjugate. If $z = a + bi$, then

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2. \ \square$$

**Example. (the Mandelbrot set)** Consider the complex function

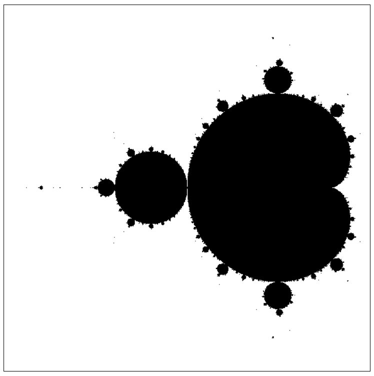
$$f(z) = z^2 + c,$$

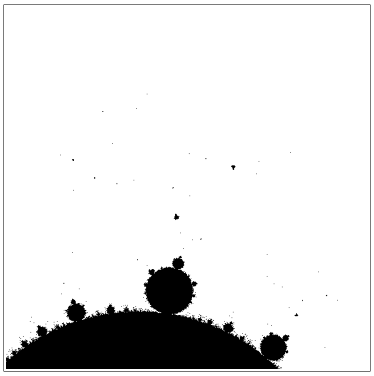
where $z$ and $c$ are complex numbers.

Pick a specific complex number $c = a + bi$. *Iterate* the function $f$ starting at $z = 0$:

$$\begin{array}{ccccc} f(0) & = & 0^2 + c & = & c \\ f(c) & = & c^2 + c & = & c^2 + c \\ f(c^2 + c) & = & (c^2 + c)^2 + c & = & c^4 + 2c^3 + c^2 + c \\ & & \vdots & & \end{array}$$

Look at the sequence of outputs $c$, $c^2 + c$, $c^4 + 2c^3 + c^2 + c$, .... If the sequence of outputs goes to infinity — that is, if the sequence gets farther and farther from the origin — then leave the point $a + bi$ "uncolored". Otherwise, "color" the point $a + bi$ black. Do this for each point $c = a + bi$ in the complex plane. The set of points that are colored black make up the **Mandelbrot set**. It is shown in the left hand picture below.





The picture on the right shows a magnified view of part of the Mandelbrot set. As you can see, it has a very complicated structure.

To code the iteration, you can write $z = x + yi$ for the current value and $c = a + bi$. Then

$$z^2 + c = (x + yi)^2 + (a + bi) = (x^2 - y^2 + a) + (2xy + b)i.$$

This gives the real and imaginary parts of the next iterate. It's easy to show that if $|x + yi| > \sqrt{2}$, then the iterates will go to infinity. So decide that if the iterates haven't exceeded $\sqrt{2}$ in length after, say, 1000 iterations, then you will assume they will never diverge — and hence, the current value of $c$ is in the Mandelbrot set.

By now, almost everyone has heard the word *fractal*, and the Mandelbrot set is probably the best known example of a fractal set. There are so many books on fractals, and so many programs available for drawing them, that it seems pointless to mention any here. □

If $z$ is a complex number, define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This series converges absolutely for all $z$.

Likewise, define

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

These series also converge absolutely for all $z$.

Now suppose $\theta$ is a *real* number. Then

$$\cos\theta + i\sin\theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}.$$

Now $i^{2n} = (i^2)^n = (-1)^n$, so

$$\cos\theta + i\sin\theta = \sum_{n=0}^{\infty} i^{2n} \frac{\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i^{2n+1} \frac{\theta^{2n+1}}{(2n+1)!} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = e^{i\theta}.$$

This is the important identity for **complex exponentials**:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

**Example.** $e^{i(2\theta)} = \cos 2\theta + i\sin 2\theta$. However,

$$e^{2i\theta} = (e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = (\cos\theta)^2 - (\sin\theta)^2 + 2\sin\theta\cos\theta \cdot i.$$

Equate real and imaginary parts:

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

These are the double angle formulas. □

**Example.** $e^{(a+b)i} = \cos(a+b) + i\sin(a+b)$. On the other hand,

$$e^{(a+b)i} = e^{ai+bi} = e^{ai}e^{bi} = (\cos a + i\sin a)(\cos b + i\sin b) = (\cos a\cos b - \sin a\sin b) + i(\sin a\cos b + \sin b\cos a).$$

Equate real and imaginary parts:

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\sin(a+b) = \sin a\cos b + \sin b\cos a$$

These are the angle addition formulas for sine and cosine. □

**Example.** You can also use complex exponentials to compute integrals. Consider the integral $\int e^x \sin 3x \, dx$. You can do this by parts.

To do it using complex exponentials, replace the sine term with the corresponding exponential.

$$\sin 3x = \operatorname{im}(\cos 3x + i \sin 3x) = \operatorname{im}(e^{3xi}),$$

so

$$\int e^x \sin 3x \, dx = \int e^x \operatorname{im}(e^{3xi}) \, dx = \operatorname{im} \int e^x e^{3xi}) \, dx = \operatorname{im} \int e^{x(1+3i)} \, dx.$$

Now

$$\int e^{x(1+3i)} \, dx = \frac{1}{1+3i} e^{x(1+3i)} + C.$$

Simplify the answer to find the imaginary part:

$$\frac{1}{1+3i} e^{x(1+3i)} = \left(\frac{1-3i}{10} e^x e^{3xi}\right) =$$

$$\left(\frac{1-3i}{10} e^x (\cos 3x + i \sin 3x)\right) = \frac{1}{10} e^x \left((\cos 3x + 3 \sin 3x) + i(-3 \cos 3x + \sin 3x)\right).$$

Therefore,

$$\operatorname{im} \int e^{x(1+3i)} \, dx = \frac{1}{10} e^x (-3 \cos 3x + \sin 3x) + C.$$

That is,

$$\int e^x \sin 3x \, dx = \frac{1}{10} e^x (-3 \cos 3x + \sin 3x) + C. \square$$