3-31-1994

Linear Systems - Repeated Roots.

The eigenvector method produces a solution to a (constant coefficient homogeneous) linear system whenever there are "enough eigenvectors". There might not be "enough eigenvectors" if the characteristic polynomial has repeated roots.

I'll consider the case of repeated roots with multiplicity two or three (i.e. double or triple roots). The general case will be handled by the method of **the exponential of a matrix**.

Suppose

$$\vec{x}\,' = A\vec{x}$$

is a linear system, λ is an eigenvalue of A of multiplicity 2, and $\vec{v}$ is an eigenvector for λ. $e^{\lambda t}\vec{v}$ is one solution; I want to find a second independent solution.

The constant coefficient equation

$$(D-3)^2 y = 0$$

had independent solutions e^{3x} and xe^{3x}.

By analogy, it's reasonable to guess a solution of the form

$$\vec{x} = te^{\lambda t}\vec{w},$$

where $\vec{w}$ is a constant vector.

Plug the guess into $\vec{x}\,' = A\vec{x}$:

$$\vec{x}\,' = te^{\lambda t}\lambda\vec{w} + e^{\lambda t}\vec{w} = A(te^{\lambda t}\vec{w}).$$

Compare terms in $te^{\lambda t}$ and $e^{\lambda t}$ on the left and right:

$$A\vec{w} = \lambda\vec{w} \quad \text{and} \quad \vec{w} = 0.$$

While it's true that $te^{\lambda t}\cdot 0 = 0$ is a solution, it's not a very *useful* solution. I'll try again, this time using

$$\vec{x} = te^{\lambda t}\vec{w}_1 + e^{\lambda t}\vec{w}_2.$$

Then

$$\vec{x}\,' = te^{\lambda t}\lambda\vec{w}_1 + e^{\lambda t}\vec{w}_1 + \lambda e^{\lambda t}\vec{w}_2.$$

Since

$$A\vec{x} = te^{\lambda t}A\vec{w}_1 + e^{\lambda t}A\vec{w}_2,$$

it follows that

$$te^{\lambda t}\lambda\vec{w}_1 + e^{\lambda t}\vec{w}_1 + \lambda e^{\lambda t}\vec{w}_2 = te^{\lambda t}A\vec{w}_1 + e^{\lambda t}A\vec{w}_2.$$

Equate coefficients in $e^{\lambda t}$, $te^{\lambda t}$:

$$A\vec{w}_1 = \lambda\vec{w}_1 \quad \text{so} \quad (A-\lambda I)\vec{w}_1 = 0,$$

$$A\vec{w}_2 = \vec{w}_1 + \lambda\vec{w}_2 \quad \text{so} \quad (A-\lambda I)\vec{w}_2 = \vec{w}_1.$$

In other words, $\vec{w}_1$ is an eigenvector, and $\vec{w}_2$ is a vector which is mapped by $A-\lambda I$ to the eigenvector. $\vec{w}_2$ is called a **generalized eigenvector**.

Example. Solve

$$\vec{x}\,' = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}\vec{x}.$$

$$\begin{vmatrix} -3-\lambda & -8 \\ 2 & 5-\lambda \end{vmatrix} = (\lambda+3)(\lambda-5)+16 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2.$$

Therefore, $\lambda = 1$ is an eigenvalue of multiplicity 2.

Now

$$A - I = \begin{bmatrix} -4 & -8 \\ 2 & 4 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a + 2b = 0$, or $a = -2b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2b \\ b \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Take $b = 1$. The eigenvector is $\langle -2, 1 \rangle$. This gives a solution

$$e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Next, I'll try to find a vector $\vec{w}$ such that

$$(A - I)\vec{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Write $\vec{w} = \langle c, d \rangle$. The equation becomes

$$\begin{bmatrix} -4 & -8 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} -4 & -8 & -2 \\ 2 & 4 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

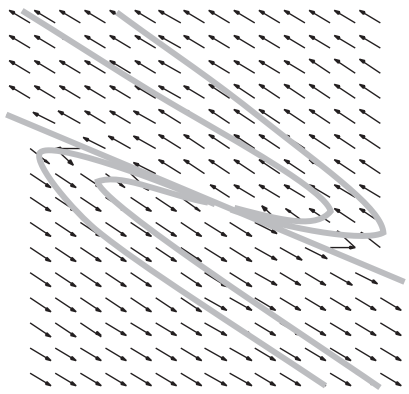
The last matrix says that $c + 2d = \frac{1}{2}$, so $c = -2d + \frac{1}{2}$. In this situation, I may take $d = 0$; doing so produces $\vec{w} = \left\langle \frac{1}{2}, 0 \right\rangle$.

This work generates the solution

$$te^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \left(te^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right).$$



The phase portrait is an improper node. $\square$

Example. Solve the system

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} \vec{x}.$$

The eigenvalues are $\lambda = 2$ and $\lambda = 1$ (double).

I'll do $\lambda = 2$ first.

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix implies that $a = 0$ and $b = 0$, so the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = c \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 1$,

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix implies that $a = 0$ and $b = c$, so the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ c \end{bmatrix} = c \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

I'll use $\vec{v} = \langle 0, 1, 1 \rangle$.

Next, I find a generalized eigenvector $\vec{w} = \langle a', b', c' \rangle$. It must satisfy

$$(A - I)\vec{w} = \vec{v}.$$

That is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solving this system yields $a' = 1$, $b' = c' + 1$. I can take $c' = 0$, so $b' = 1$, and

$$\vec{w} = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \left(t e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right). \square$$

I'll give a brief description of the situation for an eigenvalue λ of multiplicity 3. First, if there are three *independent* eigenvectors $\vec{u}$, $\vec{v}$, $\vec{w}$, the solution is

$$\vec{x} = c_1 e^{\lambda t} \vec{u} + c_2 e^{\lambda t} \vec{v} + c_2 e^{\lambda t} \vec{w}.$$

Suppose there is one independent eigenvector, say $\vec{u}$. One solution is

$$e^{\lambda t}\vec{u}.$$

Find a generalized eigenvector $\vec{v}$ by solving

$$(A - \lambda I)\vec{v} = \vec{u}.$$

A second solution is

$$te^{\lambda t}\vec{u} + e^{\lambda t}\vec{v}.$$

Next, obtain another generalized eigenvector $\vec{w}$ by solving

$$(A - \lambda I)\vec{w} = \vec{v}.$$

A third independent solution is

$$\frac{1}{2}t^2 e^{\lambda t}\vec{u} + te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}.$$

Finally, combine the solutions to obtain the general solution.
The only other possibility is that there are two independent eigenvectors $\vec{u}$ and $\vec{v}$. These give solutions

$$e^{\lambda t}\vec{u} \quad \text{and} \quad e^{\lambda t}\vec{v}.$$

Find a generalized eigenvector $\vec{w}$ by solving

$$(A - \lambda I)\vec{w} = a\vec{u} + b\vec{v}.$$

The constants a and b are chosen so that the equation is solvable.
$\vec{w}$ yields the solution

$$te^{\lambda t}(a\vec{u} + b\vec{v}) + e^{\lambda t}\vec{w}.$$

The best way of explaining why this works involves something called the **Jordan canonical form** for matrices. It's also possible to circumvent these technicalities by using the **exponential of a matrix**.