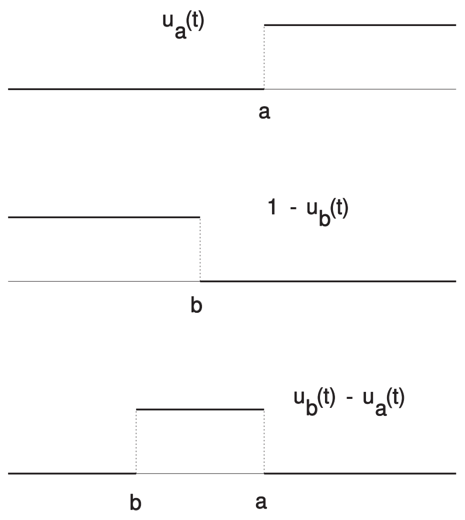
3-1-1994

Step functions.

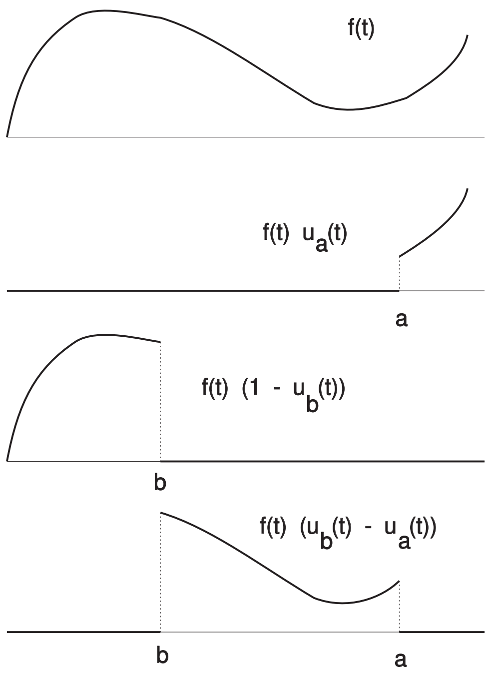
The **unit step function** or **Heaviside function** is given by

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \ge a \end{cases}$$

u_a is a "right-hand" step. You can use it to construct a "left-hand" step as well as a "square pulse". The right and left steps and the pulse are shown in the picture on the left below.



Each step is 1 unit high.



As the picture on the right shows, step functions and the pulse can be used to "cut off" chunks of functions.

- $f(t)u_a(t)$ gives the part of f lying to the right of $t = a$.
- $f(t)\,(1 - u_b(t))$ gives the part of f lying to the left of $t = b$.
- $f(t)\,(u_b(t) - u_a(t))$ gives the part of f lying between b and a.

Example. You can use step functions to express a function defined "in pieces" as a single expression.

$$f(t) = \begin{cases} 1 & \text{if } t < 0 \\ 1 - t^2 & \text{if } 0 \le t < 1 \\ t + 1 & \text{if } 1 \le t < 2 \\ 2 & \text{if } t \ge 2 \end{cases}$$

Use the "left step" for the left side, the "right step" for the right side, and the pulse for the pieces in the middle.

$$f(t) = 1 \cdot (1 - u_0(t)) + (1 - t^2) \cdot (u_0(t) - u_1(t)) + (t + 1) \cdot (u_1(t) - u_2(t)) + 2 \cdot u_2(t) =$$

$$1-t^2u_0(t)+(t^2+t)u_1(t)+(1-t)u_2(t).\ \square$$

Next, I'll compute the transforms of some step functions. With those transforms in hand, I can solve initial value problems with discontinuous forcing functions.

The problem is to compute the transform of a step function $u_a(t)$ multiplied by a function f. Doing the obvious thing

$$\int_0^\infty u_a(t)f(t)\,dt$$

does not work so well.

It turns out that it's better to use $f(t-a)$ rather than $f(t)$. This makes sense — u_a is the step shifted to a, so it's better if f is shifted to a as well. So

$$\mathcal{L}\left(u_a(t)f(t-a)\right)=\int_0^\infty u_a(t)f(t-a)e^{-st}\,dt=\int_a^\infty f(t-a)e^{-st}\,dt.$$

(Since $u_a(t)=0$ for $t<a$, it doesn't hurt to begin integrating at a.)

Now let $w=t-a$, so $dw=dt$. When $t=a$, $w=0$; as $t\to\infty$, $w\to\infty$.

$$\int_a^\infty f(t-a)e^{-st}\,dt=\int_0^\infty f(w)e^{-s(w+a)}\,dw=$$

$$e^{-sa}\int_0^\infty f(w)e^{-sw}\,dw=e^{-sa}\,\mathcal{L}\left(f(t)\right)=e^{-sa}F(s).$$

Therefore,

$$\mathcal{L}\left(u_a(t)f(t-a)\right)=e^{-sa}F(s).$$

Example.

$$\mathcal{L}\left(tu_1(t)\right)=\mathcal{L}\left((t-1+1)u_1(t)\right)=\mathcal{L}\left((t-1)u_1(t)\right)+\mathcal{L}\left(u_1(t)\right).$$

The first term is $f(t-a)u_a(t)$ for $f(t)=t$ and $a=1$. From the transform table, $\mathcal{L}\left(t\right)=\dfrac{1}{s^2}$.

The second term is $f(t-a)u_a(t)$ for $f(t)=1$ and $a=1$. From the transform table, $\mathcal{L}\left(1\right)=\dfrac{1}{s}$.

$$\mathcal{L}\left((t-1)u_1(t)\right)+\mathcal{L}\left(u_1(t)\right)=e^{-s}\frac{1}{s^2}+e^{-s}\frac{1}{s}.\ \square$$

Example. Find the transform of

$$f(t)=\begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } 0\le t<1 \\ 2-t & \text{if } 1\le t<2 \\ 0 & \text{if } t>2 \end{cases}$$

$$f(t)=0\cdot(1-u_0)+1\cdot(u_0-u_1)+(2-t)\cdot(u_1-u_2)+0\cdot u_2=u_0+(1-t)u_1+(t-2)u_2.$$

Therefore,

$$F(s)=\frac{1}{s}-\frac{1}{s^2}e^{-s}+\frac{1}{s^2}e^{-2s}.\ \square$$

Example. Compute

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s-4)}\right).$$

First, observe that

$$\frac{1}{s(s-4)} = \frac{1}{4}\left(\frac{1}{s-4} - \frac{1}{s}\right),$$

so

$$\mathcal{L}^{-1}\left(\frac{1}{s(s-4)}\right) = \frac{1}{4}\left(e^{4t} - 1\right).$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s-4)}\right) = u_2(t)\cdot\frac{1}{4}\left(e^{4(t-2)} - 1\right). \square$$

Example. $y'' + 4y' + 4y = u_2(t)$, $y(0) = 0$, $y'(0) = 2$.

Take the transform of both sides:

$$\left(-2 - 0\cdot s + s^2 Y(s)\right) + 4\left(-0 + sY(s)\right) + 4Y(s) = \frac{e^{-2s}}{s}.$$

Simplify and solve for $Y(s)$:

$$(s^2 + 4s + 4)Y(s) = 2 + \frac{e^{-2s}}{s}, \quad Y(s) = \frac{2}{(s+2)^2} + \frac{e^{-2s}}{s(s+2)^2}.$$

It's easy to invert the first term:

$$\mathcal{L}^{-1}\left(\frac{2}{(s+2)^2}\right) = 2te^{-2t}.$$

For the second term, use partial fractions:

$$\frac{1}{s(s+2)^2} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+2)^2}\right) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}.$$

I'll be lazy and write $g(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}$. Then

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s+2)^2}\right) = u_2(t)g(t-2).$$

The solution is

$$y(t) = 2te^{-2t} + u_2(t)g(t-2). \square$$

Example.

$$y'' - y = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ t & \text{if } t \ge 1 \end{cases}, \qquad y(0) = 1, \quad y'(0) = 0$$

The right side is

$$1 \cdot (u_0 - u_1) + t \cdot u_1 = u_0 + (t-1)u_1.$$

Match u_0 with $u_a \cdot f(t-a)$. Then $f(t) = 1$, so

$$\mathcal{L}(u_0) = \frac{1}{s}e^{-0 \cdot s} = \frac{1}{s}.$$

Match $(t-1)u_1$ with $u_a f(t-a)$. Since $a = 1$, $f(t-1) = t-1$, and hence $f(t) = t$. Since $\mathcal{L}(t) = \dfrac{1}{s^2}$,

$$\mathcal{L}\left((t-1)u_1\right) = \frac{1}{s^2}e^{-s}.$$

Now take the transform of both sides of the differential equation:

$$\left(-0 - s + s^2 Y(s)\right) - Y(s) = \frac{1}{s} + \frac{1}{s^2}e^{-s}.$$

Simplify and solve for $Y(s)$:

$$Y(s) = \frac{s^2+1}{s(s^2-1)} + \frac{e^{-s}}{s^2(s^2-1)}.$$

Partial fractions gives

$$\frac{s^2+1}{s(s^2-1)} = \frac{1}{s-1} - \frac{1}{s} + \frac{1}{s+1}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{s^2+1}{s(s^2-1)}\right) = e^t - 1 + e^{-t}.$$

Partial factions gives

$$\frac{1}{s^2(s^2-1)} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{2}\frac{1}{s+1}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2-1)}\right) = \frac{1}{2}e^t - t - \frac{1}{2}e^{-t}.$$

Let $g(t) = \dfrac{1}{2}e^t - t - \dfrac{1}{2}e^{-t}$. Then

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2(s^2-1)}\right) = u_1(t)g(t-1).$$

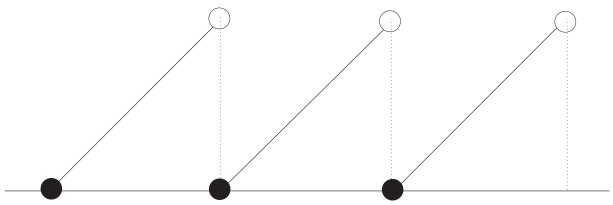
The solution is

$$y(t) = e^t - 1 + e^{-t} + u_1(t)g(t-1). \square$$

A function is **periodic with period** T if $f(t+T) = f(t)$ for all T; that is, the functions "repeats" every T units. For example, $\sin t$ and $\cos t$ are periodic with period 2π.

Here is a function with period 1 (a "sawtooth" wave):

$$f(t) = t, \quad 0 \le t < 1, \qquad f(t+1) = f(t).$$



To represent this function using step functions I'd need an infinite sum. Fortunately, there is a formula for the transform of a periodic function:

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

The idea is to break the interval $0 \le t < \infty$ up into intervals

$$0 \le t \le T, \quad T \le t \le 2T, \quad 2T \le t \le 3T, \ldots$$

On each interval, f is the same as it is on $0 \le t \le T$ by periodicity, so

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt.$$

Substitute $u = t - nT$. Then $du = dt$, and the limits $t = nT$ and $t = (n+1)T$ become $u = 0$ and $u = T$, respectively. Also,

$$e^{-st} = e^{-s(u+nT)} = e^{-snT} e^{-su}, \quad f(t) = f(u + nT) = f(u).$$

($f(u + nT) = f(u)$ is true because f repeats every T units.)
Therefore,

$$\sum_{n=0}^\infty \int_0^T e^{-snT} e^{-su} f(u)\,du = \left[\sum_{n=0}^\infty \left(e^{-sT}\right)^n\right] \int_0^T e^{-su} f(u)\,du.$$

The series is geometric with ratio e^{-sT}, so its sum is $\dfrac{1}{1 - e^{-sT}}$. Therefore,

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-su} f(u)\,du}{1 - e^{-sT}}.$$

Example. $f(t) = t$, $0 \le t < 1$, $f(t+1) = f(t)$.

The period is $T = 1$, so

$$\mathcal{L}(f(t)) = \frac{\int_0^1 e^{-st} \cdot t\,dt}{1 - e^{-s}}.$$

Integrate by parts:

$$\begin{array}{ccccc} & \dfrac{d}{dt} & & \int dt \\ + & t & & e^{-st} \\ & & \searrow & \\ - & 1 & & -\dfrac{1}{s} e^{-st} \\ & & \searrow & \\ + & 0 & \to & \dfrac{1}{s^2} e^{-st} \end{array}$$

$$\int_0^1 e^{-st} \cdot t\,dt = \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_0^1 = \frac{1}{s^2}\left(1 - e^{-s} - se^{-s}\right).$$

Therefore,

$$\mathcal{L}(f(t)) = \frac{1}{s^2} \frac{1 - e^{-s} - se^{-s}}{1 - e^{-s}}. \square$$