4-21-1994

Critical points and almost linear systems.

A system $\vec{x}\,' = \vec{F}(\vec{x})$ where the independent variable t does not appear on the right side is **autonomous**. A point $\vec{p}$ is a **critical point** for the autonomous system $\vec{x}\,' = \vec{F}(\vec{x})$ if $\vec{F}(\vec{p}) = \vec{0}$.

If $\vec{p}$ is a critical point, then

$$\frac{dx_1}{dt} = 0, \quad \frac{dx_2}{dt} = 0, \ldots.$$

Hence, $\vec{x}$ is constant — that is, the point $\vec{p}$ is a solution to the system. It is a **fixed point**, in the sense that if you start at $\vec{p}$, you never leave.

Example. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x(x-y), \\ \frac{dy}{dt} &= y^2 + 2x - y - 2. \end{aligned}$$

To find the critical points, set $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$ and solve simultaneously:

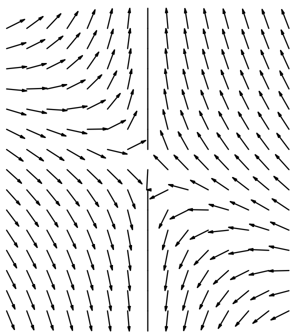
$$x(x-y) = 0$$

$$\swarrow \qquad\qquad \searrow$$

$$\begin{array}{ccccc} x = 0 & & & x = y & \\ y^2 + 2x - y - 2 = 0 & & & y^2 + 2x - y - 2 = 0 & \\ y^2 - y - 2 = 0 & & & y^2 + y - 2 = 0 & \\ (y-2)(y+1) = 0 & & & (y+2)(y-1) = 0 & \\ \downarrow & \searrow & & \downarrow & \searrow \\ y = 2 & & y = -1 & y = -2 & y = 1 \\ (0,2) & & (0,-1) & x = -2 & x = 1 \\ & & & (-2,-2) & (1,1) \end{array}$$

The critical points are $(0,2)$, $(0,-1)$, $(-2,-2)$, and $(1,1)$.

Let me give you a preview of what's coming up. Here's the direction field near the critical point $(0,2)$:



The original system was not a linear system — notice the x^2 and y^2 terms, for instance — but the picture looks unmistakably like a saddle. The idea is that near the critical points $(0,2)$, the given system is *approximately* the same as the linear system

$$\vec{x}\,' = \begin{bmatrix} -2 & 0 \\ 2 & 3 \end{bmatrix} \vec{x}.$$

For this system, det $= -6$, and the critical point *is* a saddle. Thus, you can study the behavior of a *nonlinear* system near a critical point by approximating it with a *linear* system. □

Example. The **Rössler system** is

$$\frac{dx}{dt} = -y - z, \quad \frac{dy}{dt} = x + ay, \quad \frac{dz}{dt} = b + z(x - c), \qquad a, b > 0.$$

To find the critical points, set $\dfrac{dx}{dt} = 0$, $\dfrac{dy}{dt} = 0$, and $\dfrac{dz}{dt} = 0$, and solve simultaneously.
$-y - z = 0$ gives $z = -y$, and $x + ay = 0$ gives $x = -ay$; plugging these into $b + z(x - c) = 0$ yields

$$b + (-y)(-ay - c) = 0, \quad ay^2 + cy + b = 0.$$

The roots are

$$y = \frac{-c \pm \sqrt{c^2 - 4ab}}{2a}.$$

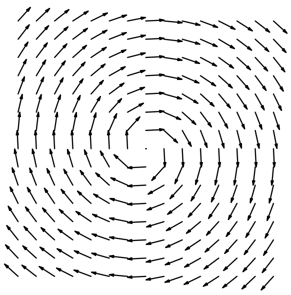
You can see that the number of critical points depends on the parameters a and b. In particular, there are two critical points if $c^2 - 4ab > 0$, there is one critical point if $c^2 - 4ab = 0$, and there are no critical points if $c^2 - 4ab < 0$.

When a system of equations models a situation in the real world, the parameters are often determined by measurement or experiment. This example shows that the qualitative behavior of the system can change in a nontrivial way as the parameters (measurements) change. Sometimes, small changes in the parameters produce very large changes in the system. □

Example. The *type* of a critical point often depends in a sensitive way on the system, even for linear systems. For example, consider the system

$$\vec{x}\,' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \vec{x}.$$

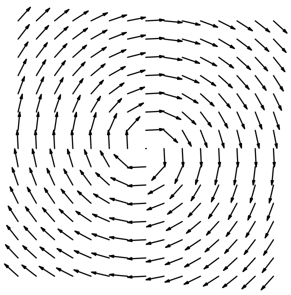
det $= 4$ and tr $= 0$, so the critical point $(0, 0)$ is a **center**, and is stable:



Suppose I change the system by a small amount:

$$\vec{x}\,' = \begin{bmatrix} 0.1 & 2 \\ -2 & 0.1 \end{bmatrix} \vec{x}.$$

det $= 4.01$ and tr $= 0.2 > 0$, so the critical point $(0,0)$ is a **spiral**, and is unstable:



Likewise, the system

$$\vec{x}\,' = \begin{bmatrix} -0.1 & 2 \\ -2 & -0.1 \end{bmatrix} \vec{x}$$

is a spiral, but this time the critical point is asymptotically stable — the solution curves spiral in toward the origin.

You can see that relatively small perturbations in a system can produce large qualitative changes. □

Example. Duffing's equation

$$\frac{d^2x}{dt^2} + x - \frac{1}{3}x^3 = A\cos(\omega t)$$

describes the behavior of a nonlinear spring with a periodic forcing term. (You can see a piece of the usual spring equation $\dfrac{d^2x}{dt^2} + x$ as well as the forcing term $A\cos(\omega t)$ in the equation.)

I'm going to demonstrate how to replace this equation with a system of first order equations. Define new variables x_1, x_2, and x_3 by

$$x_1 = x, \quad x_2 = \frac{dx}{dt}, \quad x_3 = t.$$

First,

$$\frac{dx_1}{dt} = \frac{dx}{dt} = x_2.$$

Next,

$$\frac{dx_2}{dt} = \frac{d^2x}{dt^2} = A\cos(\omega t) - x + \frac{1}{3}x^3 = A\cos(\omega x_3) - x_1 + \frac{1}{3}x_1^3.$$

Finally, $\dfrac{dx_3}{dt} = 1$. So the system is

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = A\cos(\omega x_3) - x_1 + \frac{1}{3}x_1^3, \quad \frac{dx_3}{dt} = 1.$$

Since $\dfrac{dx_3}{dt} = 1$, this system has no critical points.

Notice, incidentally, that I also removed all the t's from the right side of the system. That is, I've actually replaced the original equation with an *autonomous* system. □

I want to look more closely at the situation where a nonlinear system of differential equations can be approximated by a linear system near a critical point. For simplicity, I'll look at the case where the critical point is $(0,0)$ first.

A system

$$\vec{x}\,' = A\vec{x} + \vec{h}(\vec{x})$$

is **almost linear** at $\vec{0}$ if:

1. $\vec{0}$ is an isolated critical point of the system. (*Isolated* means that you can draw a circle centered at $\vec{0}$ which contains no other critical points.)

2. A is a constant matrix and $\det A \neq 0$. (This implies that $\vec{0}$ is a critical point of the linear system $\vec{x}\,' = A\vec{x}$.)

3. $\vec{h}(\vec{x})$ has continuous first partial derivatives and

$$\lim_{|\vec{x}|\to 0} \frac{|\vec{h}(\vec{x})|}{\vec{x}} = 0.$$

The third condition means, roughly, that $\vec{h}(\vec{x})$ is of quadratic order or higher near the critical point $\vec{0}$. This ensures that the linear system $\vec{x}\,' = A\vec{x}$ is a good approximation to the original system near $\vec{0}$.

In many cases, it's easy to find the matrix A and also to see that the original system is almost linear. I'll consider the case of a 2 variable system. Suppose the system is

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y).$$

If f and g have continuous *second* order partial derivatives at a critical point (a,b), then the system is almost linear at (a,b). I can write down the (2 variable) Taylor expansions for f and g:

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f_2(x,y),$$

$$g(x,y) = g(a,b) + \frac{\partial g}{\partial x}(a,b)(x-a) + \frac{\partial g}{\partial y}(a,b)(y-b) + g_2(x,y).$$

The higher order terms satisfy

$$\lim_{|\vec{x}|\to 0} \frac{|f_2(x,y)|}{|\vec{x}|} = 0, \quad \lim_{|\vec{x}|\to 0} \frac{|g_2(x,y)|}{|\vec{x}|} = 0.$$

Since (a,b) is a critical point, $f(a,b) = g(a,b) = 0$. Thus,

$$f(x,y) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f_2(x,y;a,b),$$

$$g(x,y) = \frac{\partial g}{\partial x}(a,b)(x-a) + \frac{\partial g}{\partial y}(a,b)(y-b) + g_2(x,y;a,b).$$

The system is

$$\frac{d}{dt}\begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} f_2(x,y) \\ g_2(x,y) \end{bmatrix}.$$

Assuming that the matrix of first partials has nonzero determinant, this shows that the system is almost linear at (a,b). The corresponding linear system is

$$\frac{d}{dt}\begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}.$$

Example. Show directly that the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x^3 - xy^2 \\ -x^2 y - y^3 \end{bmatrix}$$

is almost linear at $(0, 0)$. Construct the correspond linear system and use it to determine the behavior of the system near $(0, 0)$.

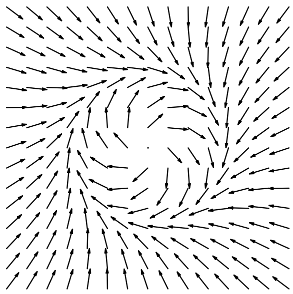
I need to verify that the components of $\begin{bmatrix} -x^3 - xy^2 \\ -x^2 y - y^3 \end{bmatrix}$ are negligible near $\vec{0}$. Now $|\vec{x}| = \sqrt{x^2 + y^2} = r$, so I convert to polar and compute the limits:

$$\lim_{r \to 0} \frac{|-x^3 - xy^2|}{r} = \lim_{r \to 0} \frac{|r^3 \cos\theta|}{r} = \lim_{r \to 0} r^2 |\cos\theta| = 0,$$

$$\lim_{r \to 0} \frac{|-x^2 y - y^3|}{r} = \lim_{r \to 0} \frac{|r^3 \sin\theta|}{r} = \lim_{r \to 0} r^2 |\sin\theta| = 0.$$

The corresponding linear system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$



$\det = 2$ and $\operatorname{tr} = 2$, so $\det > \dfrac{1}{4} \operatorname{tr}^2$. The critical point is a spiral; since $\operatorname{tr} > 0$, it is unstable (a source). □

Example. Suppose x and y represent the populations of two species in an ecosystem. Without any interaction, it's reasonable to model the growth of x and of y by the logistic equation:

$$\frac{dx}{dt} = x(a - bx), \quad \frac{dy}{dt} = y(c - dy).$$

If the species interact in a competitive way, I adjust the growth rates downward by a factor which measures how often they interact. It's reasonable to take the frequency of interaction to be proportional to the product of the two populations xy; the adjusted equations are

$$\frac{dx}{dt} = x(a - bx) - pxy = x(a - bx - py), \quad \frac{dy}{dt} = y(c - dy) - qxy = y(c - dy - qx).$$

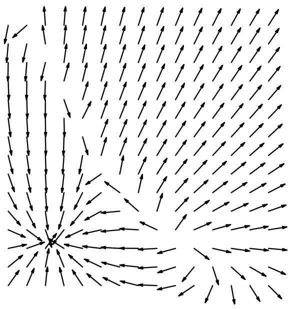
Here's an example with some made-up numbers. The signs on the numbers are wrong for a model of the real world, so this example merely illustrates the mathematics.

I'll locate the critical points for the system

$$\frac{dx}{dt} = x(x+y-3), \quad \frac{dy}{dt} = y(2x+y-4).$$

For each critical point, I will then find and classify the corresponding linear system.

Here's the field for the original system:



Find the critical points:

$$\begin{array}{ccccccc}
 & & & x(x+y-3)=0 & & & \\
 & \swarrow & & & \searrow & & \\
x=0 & & & & & y=3-x & \\
y(2x+y-4)=0 & & & & & y(2x+y-4)=0 & \\
y(y-4)=0 & & & & & (3-x)(x-1)=0 & \\
\downarrow & \searrow & & & & \downarrow & \searrow \\
y=0 & & y=4 & & & x=3 & x=1 \\
(0,0) & & (0,4) & & & y=0 & y=2 \\
 & & & & & (3,0) & (1,2)
\end{array}$$

The critical points are $(0,0)$, $(0,4)$, $(3,0)$, and $(1,2)$. For future reference, note that the matrix of first partials is

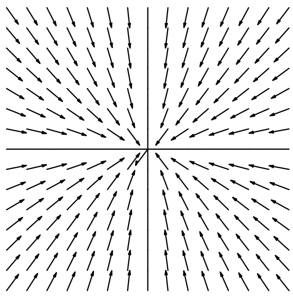
$$A = \begin{bmatrix} 2x+y-3 & x \\ 2y & 2x+2y-4 \end{bmatrix}.$$

I will be plugging the critical points into this matrix to find the corresponding linear systems.

At $(0,0)$,

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}.$$

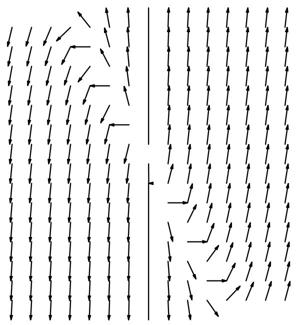
$\det = 12$ and $\operatorname{tr} = -7$. Since $\det < \dfrac{1}{4}\operatorname{tr}^2$, the critical point is an improper node; since $\operatorname{tr} < 0$, it is asymptotically stable. You can verify that -3 and -4 are the eigenvalues, and that $\langle 1,0\rangle$ and $\langle 0,1\rangle$ are corresponding eigenvectors. Here's the picture for $-0.2 < x < 0.2$ and $-0.2 < y < 0.2$:



At $(0,4)$,

$$A = \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix}.$$

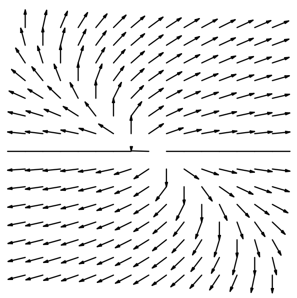
$\det = 4$ and $\operatorname{tr} = 5$. Since $\det < \frac{1}{4}\operatorname{tr}^2$, the critical point is an improper node; since $\operatorname{tr} > 0$, it is unstable. You can verify that 1 and 4 are the eigenvalues, and that $\langle -3, 8\rangle$ and $\langle 0, 1\rangle$ are corresponding eigenvectors. Here's the picture for $-0.2 < x < 0.2$ and $3.8 < y < 4.2$:



At $(3,0)$,

$$A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}.$$

$\det = 6$ and $\operatorname{tr} = 5$. Since $\det < \frac{1}{4}\operatorname{tr}^2$, the critical point is an improper node; since $\operatorname{tr} > 0$, it is unstable. You can verify that 3 and 2 are the eigenvalues, and that $\langle 1, 0\rangle$ and $\langle 3, -1\rangle$ are corresponding eigenvectors. Here's the picture for $2.8 < x < 3.2$ and $-0.2 < y < 0.2$:

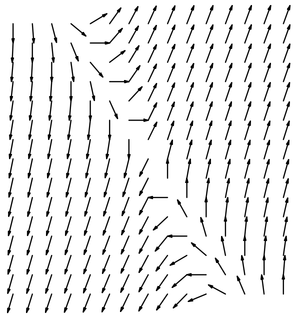


Finally, at $(1,2)$,

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}.$$

$\det = -2$ and $\operatorname{tr} = 3$. Since $\det < 0$, the critical point is a saddle, and it's unstable. You can verify the eigenvalues ar $\frac{3}{2} \pm \frac{\sqrt{17}}{2}$, and that $\left\langle 1, \frac{1}{2} + \frac{\sqrt{17}}{2}\right\rangle$ and $\left\langle 1, \frac{1}{2} - \frac{\sqrt{17}}{2}\right\rangle$ are corresponding eigenvectors.

Here's the picture for $0.8 < x < 1.2$ and $1.8 < y < 2.2$:



To see how these fit together, paste little copies of the four critical point fields onto the big field picture at the start of the problem. $\square$