10-22-1993

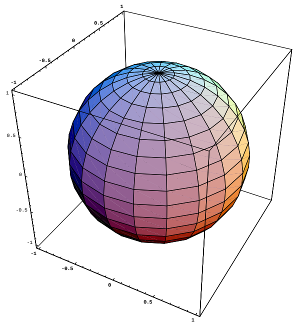
The Divergence Theorem.

The **Divergence Theorem** relates flux of a vector field through the boundary of a region to a triple integral over the region. In particular, let $\vec{F}$ be a vector field, and let R be a region in space. Then

$$\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} = \int\!\!\int\!\!\int_{R} \operatorname{div} \vec{F}\, dV.$$

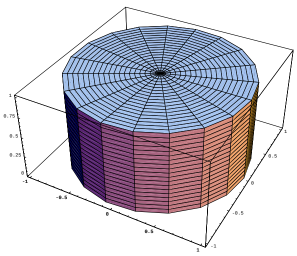
Here are some examples which should clarify what I mean by the *boundary* of a region.

Examples. If R is the solid sphere $x^2 + y^2 + z^2 \leq 1$, its boundary ∂R is the sphere $x^2 + y^2 + z^2 = 1$.

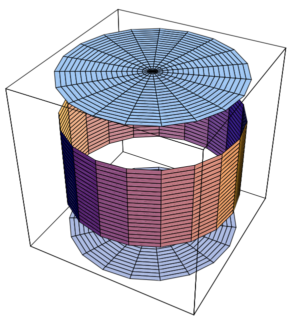


Suppose R is the solid cylinder

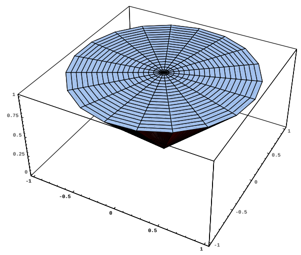
$$x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1.$$



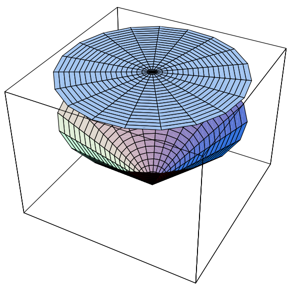
The boundary ∂R consists of the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 1$, the top $x^2 + y^2 \leq 1$, $z = 1$, and the bottom $x^2 + y^2 \leq 1$, $z = 0$.



Suppose R is the solid cone $\sqrt{x^2 + y^2} \le 1$.



The boundary ∂R consists of the top $x^2 + y^2 \le 1$, $z = 1$ and the conical surface $z = \sqrt{x^2 + y^2}$ for $0 \le z \le 1$.



□

Here are some examples which show how the Divergence Theorem is used.

Example. If $\vec{F} = \langle x, y, z \rangle$ is the radial vector field and R is a region in space, then

$$\operatorname{div} \vec{F} = 1 + 1 + 1 = 3.$$

The Divergence Theorem says

$$\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} = \int\!\!\int\!\!\int_{R} 3\, dV = 3 \cdot (\text{the volume of R}).$$

This is similar to the formula for the area of a region in the plane which I derived using Green's theorem. □

Example. Here is an interpretation of div $\vec{F}$ which is based on the Divergence Theorem. Construct a *small* solid sphere R centered at the point P. If div $\vec{F} = 0$ at P, then by the Divergence Theorem

$$\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} \approx 0.$$

That is, there is approximately no net flux out through the boundary of the sphere. Likewise, if div $\vec{F} > 0$, then $\displaystyle\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} > 0$ — there is net flux *out* through the boundary. And if div $\vec{F} < 0$, then $\displaystyle\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} < 0$ — there is net flux *in* through the boundary. □

Example. The flux of $\operatorname{curl} \vec{F}$ through the boundary of a region is 0. For if R is a region,

$$\int\!\!\int_{\partial R} \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int\!\!\int\!\!\int_{R} \operatorname{div} \operatorname{curl} \vec{F} \, dV = 0,$$

because $\operatorname{div} \operatorname{curl} \vec{F} = 0$.

This is analogous to the result that the line integral of a gradient field around a closed curve is 0. □

Example. Find the flux of $\vec{F} = \langle x^2, y^2, -z^2 \rangle$ out through the surface of the cube

$$0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1.$$

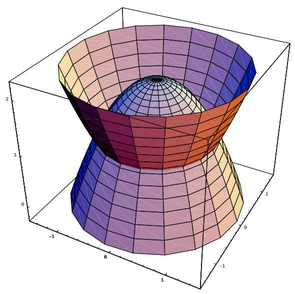
Let R denote that cube and ∂R its boundary, oriented with the outward normal. $\operatorname{div} \vec{F} = 2x + 2y - 2z$, so by the Divergence Theorem the flux is

$$\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} = \int_0^1 \int_0^1 \int_0^1 = (2x + 2y - 2z) \, dx \, dy \, dz = 1.$$

The alternative would be to compute the flux through each of the 6 sides and add up the results. □

Example. Find the flux of $\vec{F} = \langle x^2 y + 2x, 3 - x^2 y, x^2 z - 2xyz \rangle$ outward through the surface of the region R bounded by the paraboloids

$$z = x^2 + y^2 \text{ and } z = 2 - x^2 - y^2.$$



□

$$\operatorname{div} \vec{F} = 2xy + 2 - x^2 + x^2 - 2xy = 2.$$

The paraboloids intersect in $x^2 + y^2 = 1$, so the projection of R into the x-y plane is the unit disk $x^2 + y^2 \le 1$. Convert to cylindrical; the region is

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad r^2 \le z \le 2 - r^2.$$

By the Divergence Theorem, the flux is

$$\int\!\!\int_{\partial R} \vec{F} \cdot \overrightarrow{dS} = \int_0^1 \int_0^{2\pi} \int_{r^2}^{2-r^2} 2r \, dz \, d\theta \, dr = 2\pi \approx 6.28319. \ \square$$

Example. Find the flux of the unit radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|}$ outward through the surface of the sphere $x^2 + y^2 + z^2 \le a^2$.

Let R be the solid sphere $x^2+y^2+z^2 \le a^2$. By the Divergence Theorem,

$$\iint_{\partial R} \vec{F} \cdot \overrightarrow{dS} = \iiint_R \operatorname{div} \vec{F} dV.$$

Convert to spherical; the region is

$$0 \le \rho \le a, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

Now $\vec{F} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\langle x, y, z\rangle$, so

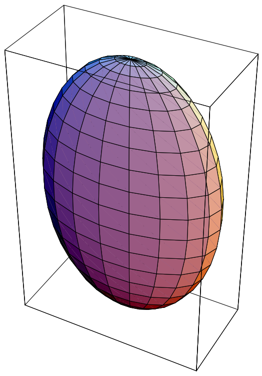
$$\operatorname{div} \vec{F} = \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{\rho}.$$

Hence,

$$\iiint_R \operatorname{div} \vec{F} dV = \int_0^{2\pi} \int_0^{\pi} \int_0^a \frac{2}{\rho}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 4\pi a^2. \quad \square$$

Example. Use the Divergence Theorem to find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$



$\square$

I can parametrize the ellipsoid by

$$x = a \sin p \cos t, \quad y = b \sin p \sin t, \quad z = c \cos p, \qquad 0 \le p \le \pi, \quad 0 \le t \le 2\pi.$$

(This is similar to spherical coordinates, if you think about it.)
The normal is

$$\overrightarrow{T_p} \times \overrightarrow{T_t} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a\cos p \cos t & b \cos p \sin t & -c \sin p \\ -a \sin p \sin t & b \sin p \cos t & 0 \end{vmatrix} = \langle bc(\sin p)^2 \cos t, ac(\sin p)^2 \sin t, ab \sin p \cos p\rangle.$$

I use the radial field:

$$\vec{F} = \langle x, y, z\rangle = \langle a \sin p \cos t, b \sin p \sin t, c \cos p\rangle.$$

Then

$$\vec{F} \cdot (\overrightarrow{T_p} \times \overrightarrow{T_t}) = abc \sin p.$$

By symmetry, I can take the flux out of the part in the first octant and multiply by 8:

$$8 \int_0^{\pi/2} \int_0^{\pi/2} abc \sin p \, dp \, dt = 4\pi abc.$$

By the Divergence Theorem, this is 3 times the volume, so the volume is $\dfrac{4}{3}\pi abc$. $\square$