# Uniqueness of Parisi's Scheme for Replica Symmetry Breaking

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**Abstract**

Replica symmetry breaking in spin glass models is investigated using elements of the theory of permutation groups. It is shown how the various types of symmetry breaking give rise to special algebras and that Parisi's scheme may be uniquely characterized by two simple conditions on these algebras, namely transposition symmetry and simple extensibility. An alternative to the Parisi scheme is shown to be unacceptable.

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# 1 Introduction

In 1979 Parisi proposed a scheme for replica symmetry breaking which led to a continuously distributed set of order parameters $\{q(x), 0 < x < 1\}$ for the replica symmetry breaking phases of spin glasses [Parisi (1979a)]. This breakthrough came at the end of a number of investigations into the possible forms of replica symmetry breaking [Almeida and Thouless (1978), Blandin (1978), Bray and Moore (1978), Parisi (1979b)]. Although Parisi acknowledged [Parisi (1980)] that his scheme was justified mainly by its ability to reproduce numerical simulations, no further theoretical justification for the scheme seems to have been brought forward since then (see for example reviews in [Binder and Young (1986), Mezard et al (1987), Fischer and Hertz (1991)]).

In this paper certain aspects of the theory of finite permutation groups is used to investigate the question of uniqueness of the Parisi prescription for replica symmetry breaking. The theory of finite groups is a vast subject with most of the modern literature formulated in almost impenetrable mathematical notation. The older text books [Burnside (1911), Carmichael (1937)] and some of the modern [Gorenstein (1968), Kargapolov (1979)] are more accessible. For the present study the most useful presentation of the theory of finite permutation groups is found in [Wielandt (1964)] and in the discussion of wreath products by [Kerber (1971)].

Group theoretical concepts such as single and double transitivity, transposition symmetry and primitivity are shown to be of major significance for the analysis. The most important property is, however, that of *extensibility* which roughly speaking means that a model for replica symmetry breaking valid for a given finite set of replicas can be extended to a model valid for larger finite set of replicas without changing the structure of the original model. In the simplest case this property leads uniquely to the Parisi scheme.

A preliminary account of the results of this paper was previously given in a set of lecture notes [Lautrup (1987)]. The level of the present paper is aimed at readers that are not experts in group theory and the paper is therefore reasonably self-contained. The presentation is, however, not complete in the mathematical sense, but introduces only those concepts and proves only those results that are necessary for understanding the arguments.

In section 2 replica theory is reviewed and presented in order to establish notation. In section 3 we introduce permutations and discuss in section 4 the relation between symmetry breaking solutions and permutation groups. In section 5 the concept of simple transitivity is introduced, and after a discussion of transposition symmetry and the reasons for it in section 6, the important concept of double transitivity is defined in section 7. In section 8 an algebraic basis for double transitivity is established, and in section 9 the concept of extensibility is defined in terms of this algebra and shown to lead to the Parisi scheme. In section 10 we analyze the consequences of primitivity and consider alternative schemes (elaborated in appendix A). Finally, section 11 contains the conclusions.

## 2 Replica theory

The use of the replica method in the theory of spin glasses goes back to the famous paper by [Edwards and Anderson (1975)]. Generically, one considers a system with $N$ components (spins) described by a state vector $S$. The energy of the system $E(S,J)$ is characterized by a set of parameters $J$ and in the canonical SK-model $E(S,J) = -\sum_{i<j} J_{ij} S_i S_j$ [Sherrington and Kirkpatrick (1975), Kirkpatrick and Sherrington (1978)]. The thermodynamics of such a system in equilibrium with a heat bath is described by the partition function $Z(J) = \sum_S \exp(-\beta E(S,J))$ where $\beta = 1/T$ is the inverse temperature. The free energy per spin is defined as $f(J) = -\frac{T}{N} \log Z(J)$ and is supposed to take a definite value in the thermodynamic limit $N \to \infty$.

The values of the parameters $J$ are assumed to be subject to noise and distributed according so some probability distribution $P(J)$, and we denote averages over this distribution by $\ll \cdots \gg$. In the SK-model the noise is uncorrelated uniformly Gaussian with mean $\ll J_{ij} \gg = \mu/N$ and variance $\ll (J_{ij} - \mu)^2 \gg = \sigma^2/N$. The quantity determining the average thermodynamics of the system is consequently the average free energy $f = \ll f(J) \gg$. The calculation of the average presents a mathematical problem because of the logarithm. It was proposed by [Edwards and Anderson (1975)] to use the formal trick of writing $\ll \log Z(J) \gg = \lim_{n\to 0} (\ll Z(J)^n \gg -1)/n$ thus recasting the problem in the form of determining an analytic expression in $n$ for the average $\ll Z(J)^n \gg$ in the limit of $n \to 0$.

For integer $n > 0$ it is easy to construct an expression for this quantity

$$Z(J)^n = \left( \prod_{a=1}^{n} \sum_{S_a} \right) e^{-\beta \sum_{a=1}^{n} E(S_a, J)} \tag{1}$$

Physically this is the partition function for a system of $n$ identical non-interacting replicas of the original system in contact with the same heat bath. The averaging over the noise in the couplings between the spins leads, however, in the end to an effective coupling between the replicas.

In the SK-model it is possible in the thermodynamic limit to derive a supposedly exact mean field theory for the replicated system (see also [Amit et al (1987)]). The free energy becomes for finite integer $n$

$$f(m,q) = -\frac{\beta\sigma^2}{4} + \frac{\mu}{2n} \sum_a m_a^2 + \frac{\beta\sigma^2}{2n} \sum_{a<b} q_{ab}^2 - \frac{T}{n} \log Z(m,q) \tag{2}$$

where $Z(m,q)$ is the following partition function for a single replicated spin

$$Z(m,q) = \left( \prod_a \sum_{S_a} \right) e^{\beta\mu \sum_a m_a S_a + \beta^2\sigma^2 \sum_{a<b} q_{ab} S_a S_b} \tag{3}$$

and where $m_a$ and $q_{ab}$ are thermodynamic (order) parameters. Notice that we have taken the limit $N \to \infty$ for arbitrary finite $n$ whereas a strict application of the replica trick would require that $n \to 0$ first.

The order parameters $m_a$ and $q_{ab}$ may be interpreted as the average magnetization in the replicas and the average state vector overlap between replicas. Although the matrix elements of $q$ are used only for $a < b$ we extend the matrix naturally by requiring it to be symmetric (and setting the diagonal elements to 0). With this extension it is clear that *the free energy is invariant under any permutation of the replicas.*

The physical values of the order parameters are determined by the requirement that the free energy be extremal. This leads to the following equations of stationarity (also called the mean field equations)

$$\left.\begin{array}{rcl} m_a & = & < S_a > \\ q_{ab} & = & < S_a S_b > \end{array}\right\} \tag{4}$$

where the brackets $< \cdots >$ refer to averages over the system defined by the single spin partition function (3). The stability of the solutions may be determined from the second derivatives of the free energy.

The symmetry of the free energy with respect to all permutations does not imply that the solutions to the mean field equations are also symmetric. They may spontaneously break the permutation symmetry. Nevertheless the first and most natural guess for the form of the solutions is that it is *replica symmetric*, *i.e.* $m_a = m$ and $q_{ab} = q$. In that case the free energy becomes an analytic function of $n$ which may be evaluated in the limit of $n \to 0$ [Sherrington and Kirkpatrick (1975)]. When the stability matrix is also be evaluated around the symmetric solution the result is [Almeida and Thouless (1978)] for $m_a = 0$ that the replica symmetric spin glass solution is everywhere *unstable*.

The instability of the symmetric spin glass solution necessitates the introduction of broken replica symmetry. Various special forms that this breakdown can take may be guessed [Bray and Moore (1978)], but it was Parisi who first constructed a general formalism for the breakdown [Parisi (1979a)]. It was based on a special *ansatz* for the form of $q_{ab}$ which in the end permitted the approach to the limit $n \to 0$.

Let $m_i$, $i = 1, \ldots, I+1$ be a set of integers (not to be confused with $m_a$) such that $m_1 = 1$ and $m_{I+1} = n$ and such that the ratios $m_{i+1}/m_i$ are all integers greater than 1. If we denote the integer quotient of two integers by $[x/y]$ then the Parisi *ansatz* takes the form (when the replicas are numbered from 0 to $n-1$)

$$q_{ab} = q_i \quad \text{if } [a/m_i] \neq [b/m_i] \text{ but } [a/m_{i+1}] = [b/m_{i+1}] \tag{5}$$

where the $q_i$ are real numbers. In this way one may characterize the breakdown of replica symmetry in the spin glass phase by a set real numbers $q_i$. Defining the piecewise constant function $q(x) = q_i$ for $m_i < x < m_{i+1}$ on the interval $1 < x < n$, then the 'limit' of $n \to 0$ which is approached through $I \to \infty$ this function 'becomes' a continuous function on the unit interval $0 < x < 1$.

# 3 Permutations

A *permutation* $\pi$ is a mapping $a \to \pi(a)$ of the set $A$ of $n = |A|$ replicas onto itself. The group of all permutations of $n$ objects is called the *symmetric group* of degree $n$ and denoted $\mathcal{S}_n$. The number of group elements is called the order of the group and is $n!$ for $\mathcal{S}_n$. The *alternating* group consisting of all even permutations is denoted $\mathcal{A}_n$ and is of order $n!/2$.

In the following we use standard mathematical notation and write permutations as products of independent cycles. A cycle of length $k$ is denoted $(a_1 a_2 a_3 \cdots a_k)$ and corresponds to an operation whereby $a_1$ is replaced by $a_2$, $a_2$ by $a_3$, and so on until $a_k$ is replaced by $a_1$. Trivial cycles of length one may be omitted. The identity element $(1)(2)\cdots(n)$ is denoted 1.

The first non-trivial symmetric group is $\mathcal{S}_2 = \{1, (12)\}$. It has two elements: the identity and a simple transposition, but contains no proper subgroups. The first group containing proper non-trivial subgroups is $\mathcal{S}_3 = \{1, (12), (23), (31), (123), (321)\}$. This group has three subgroups, each isomorphic to $\mathcal{S}_2$, consisting of the identity and one of the cycles of length 2 and one further subgroup consisting of the identity and the two cycles of length 3. We shall take a number of examples from $\mathcal{S}_4$ which has 24 elements falling into 5 classes with different cycle structure. They are: 1) the unit 1, 2) the 6 transpositions $(ab)$, 3) the 3 double transpositions $(ab)(cd)$, 4) the 8 cycles $(abc)$ of length 3, and 5) the 6 cycles $(abcd)$ of length 4.

The *fundamental representation* of the symmetric group associates an $n \times n$ matrix $g = g(\pi)$ defined by $g_{ab} = \delta_{a\pi(b)}$ with any permutation $\pi$. The representation is faithful, orthogonal ($g^\top g = 1$), and behaves in every respect like the symmetric group itself.

Every abstract finite group is isomorphic to at least one permutation group, because the abstract group multiplication table simply expresses how each element permutes all elements among each other. But there are actually many faithful ways for an abstract group to be represented by permutations. Most of the properties we shall discuss in the following, for example transitivity, are specific to permutations and not in general valid for abstract finite groups.

# 4 Invariance

Let $(m, q)$ be a set of values of the order parameters for a replicated spin glass and let $\pi$ be an arbitrary permutation. We then define the transformed order parameters to be

$$\begin{aligned} m'_a &= m_{\pi(a)} \\ q'_{ab} &= q_{\pi(a)\pi(b)} \end{aligned}$$

The invariance of the free energy (2) means that it takes the same value before and after the transformation: $f(m', q') = f(m, q)$. This of course implies that if $(m, q)$ is a solution to the mean field equations (4), so will every permuted set $(m', q')$.

In the general case some permutations will leave the solution unchanged whereas others will change it. If all permutations leave the solution invariant, it must be replica symmetric, *i.e.* $m_a = m$ and $q_{ab} = q$. Otherwise the solution is said to break the replica symmetry spontaneously (*i.e.* without external provocation). The set of permutations $G$ that actually leave a given solution invariant must satisfy

$$\left.\begin{array}{rcl} m_a & = & m_{\pi(a)} \\ \\ q_{ab} & = & q_{\pi(a)\pi(b)} \end{array}\right\} \tag{6}$$

for all $\pi \in G$. In terms of the fundamental representation we have $m = gm$ and $g^\top q g = q$ for every $g \in G$. This set of permutations obviously constitutes a subgroup of the symmetric group $\mathcal{S}_n$.

Every solution to the mean field equations may be characterized by the corresponding invariance group. The replica symmetric solutions are for example invariant under the full symmetric group. Conversely, for any subgroup $G$ of the symmetric group there will be a class of solutions to the mean field equations which is invariant under this group. As the sum or product of invariant matrices is again invariant, the class of overlap matrices $q$ corresponding to a given invariance group forms a *ring*, called the *centralizer ring* of $G$, consisting of all those matrices $q$ that commute with all elements $g$ of the group.

Consider for example the subgroup $\{1, (12)\}$ of $\mathcal{S}_3$. The most general solution to the mean field equations invariant under this group has $m_1 = m_2 = m$ and an overlap matrix $q$ of the form

$$\begin{pmatrix} q_0 & q_1 & q_2 \\ q_1 & q_0 & q_2 \\ q_3 & q_3 & q_4 \end{pmatrix} \quad \text{or just} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 3 & 3 & 4 \end{pmatrix} \tag{7}$$

where $m$ and $q_i$ are arbitrary real parameters. This matrix is generically asymmetric and we need to choose $q_3 = q_2$ (in shorthand 3=2) in order to enforce the symmetry. Notice that when displaying the structure of an overlap matrix it is sufficient and much clearer just to write down the indices of different parameters.

Another example is the subgroup $\mathcal{A}_3 = \{1, (123), (321)\}$ of $\mathcal{S}_3$. Here we have $m_1 = m_2 = m_3 = m$, and the most general invariant $q$-matrix takes the form

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \tag{8}$$

In this case enforcement of the symmetry implies that $2 = 1$ and consequently the matrix becomes completely replica symmetric, *i.e.* invariant under $\mathcal{S}_3$.

# 5 Simple transitivity

One of the important concepts in the theory of permutation groups is that of *transitivity*. Two replicas $a$ and $b$ are said to be *connected* in the invariance group $G$, if there is a permutation $\pi \in G$ such that $b = \pi(a)$. Obviously this relation of connectedness is an equivalence relation, which divides the set of replicas $A$ into $k$ disjoint equivalence classes, $A_1, A_2, \ldots, A_k$, called *orbits*. Inside each orbit the replicas are connected, whereas there is no connection between replicas belonging to different orbits. If there is only one orbit (*i.e.* $k = 1$) the invariance group $G$ is said to be (singly or simply) *transitive*, otherwise it is *intransitive*. In the two examples above the invariance group of (7) is intransitive whereas that of (8) is transitive.

## 5.1 Intransitive groups

In an intransitive group every permutation acts separately on each orbit, such that we can write $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ where $\pi_i$ is a permutation acting on $A_i$ only. These permutations form by themselves a transitive group $G_i$ of degree $|A_i|$, called the transitive *projection* of $G$ on $A_i$. The original group $G$ is therefore equal (or isomorphic) to a subgroup of the direct product of each of its transitive projections, $G \subseteq G_1 \otimes G_2 \otimes \ldots \otimes G_k$, also called a *subdirect* product. Consider for example the group $G = \{1, (12)(34)\}$ of degree 4 which is intransitive with two orbits $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$. The corresponding transitive projections $G_1 = \{1, (12)\}$ and $G_2 = \{1, (34)\}$ are in both cases isomorphic to the symmetric group $\mathcal{S}_2$. The product $G_1 \otimes G_2 = \{1, (12), (34), (12)(34)\}$ is isomorphic to $\mathcal{S}_2 \otimes \mathcal{S}_2$ and contains $G$ as a subgroup.

From the invariance property (6) we see that all replicas in the same orbit have the same magnetization, *i.e.* $m_a = m_i$ for $a \in A_i$ where the $m_i$ are arbitrary parameters. This is the most general form of the magnetization for an intransitive invariance group with $k$ transitive projections. Notice that the direct product of the projections leads to the same result.

Similarly the overlap matrix $q$ decomposes into $k^2$ submatrices, $q^{ij}$, corresponding to the direct products of two orbits $A_i \otimes A_j$. Each diagonal submatrix $q^{ii}$ is invariant under the transitive projection $G_i$, whereas the off-diagonal (generally non-square) submatrix $q_{ij}$ with $i \neq j$ is invariant under a subgroup of the direct product $G_i \otimes G_j$. If the off-diagonal matrix is invariant under the direct product itself then all matrix elements must be equal because all combinations of permutations on $A_i$ with permutations on $A_j$ will be allowed. This is illustrated by $G = \{1, (12)(34)\} \subset G_1 \otimes G_2$ for which the overlap matrix takes the asymmetric block form

$$\left(\begin{array}{cc|cc} 0 & 1 & 4 & 5 \\ 1 & 0 & 5 & 4 \\ \hline 6 & 7 & 2 & 3 \\ 7 & 6 & 3 & 2 \end{array}\right) \tag{9}$$

If instead $G = G_1 \otimes G_2$ we must also have $5 = 4$ and $7 = 6$.

## 5.2 Transitive groups

For a transitive group we have $m_a = m$ for all $a$. Similarly the diagonal elements $q_{aa}$ are all equal. Notice that transitivity does not mean that all the replica overlaps, $q_{ab}$, are equal, only that each row (or column) is a permutation of any other row (or column). This is illustrated by the permutation group $\{1, (12)(34), (13)(24), (14)(23)\}$ of degree 4, which is clearly transitive. The overlap matrix takes the form

$$\left(\begin{array}{cc|cc} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ \hline 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right) \tag{10}$$

In this case the invariance group forces the matrix to be symmetric in contrast with the intransitive case (9).

Parisi used the condition that the sum over each row, $\sum_b q_{ab}$, be independent of the row index $a$ to guide the derivation of the form of his *ansatz* for replica symmetry breaking [Parisi (1980)]. Since simple transitivity implies that each row is a permutation of any other, his condition is a consequence of transitivity.

# 6 Transposition symmetry

In the preceding sections the question of the symmetry of the overlap matrix under transposition of rows and columns has surfaced several times. In the examples (7), (8) and (9) the generic overlap matrix is asymmetric whereas in (10) it is symmetric. It is also clear that it is not the transitivity that enforces symmetry, because (8) and (10) are both transitive. In the example (8) we also saw how the requirement of symmetry changed the actual invariance group.

We shall now investigate the consequences of demanding that the invariance group enforces transposition symmetry of the overlap matrix. Such a group will be called a *symmetrizing group* and should be distinguished from *the* symmetric group $\mathcal{S}_n$ containing all permutations of $n$ objects.

The necessary and sufficient condition for an invariance group $G$ to be symmetrizing is that to every pair of replicas $a$ and $b$ there must exist a permutation $\pi \in G$, which *transposes* the pair, *i.e.* for which $\pi(a) = b$ and $\pi(b) = a$. Thus the permutation must be of the form $\pi = (ab)\rho$ with $\rho$ neither affecting $a$ nor $b$.

The group generated by all transpositions is uninteresting and trivially equal to $\mathcal{S}_n$ because every permutation can be written as a product of transpositions. The simplest example of a non-trivial symmetrizing invariance group is furnished by the group $\{1, (12)(34), (13)(24), (14)(23)\}$ giving rise to (10). Another is the 4'th degree group $\{1, (12), (34), (12)(34), (14)(23), (31)(42), (1324), (3142)\}$ which has an overlap matrix of the form

$$
\left(\begin{array}{cc|cc}
0 & 1 & 2 & 2 \\
1 & 0 & 2 & 2 \\
\hline
2 & 2 & 0 & 1 \\
2 & 2 & 1 & 0
\end{array}\right) \tag{11}
$$

It follows immediately that *a symmetrizing invariance group is singly transitive* because for any pair of replicas a transposing permutation will connect the replicas. Whereas a symmetrizing group is transitive, the opposite is, however, not the case as demonstrated by (8). On the other hand, we may conclude that an intransitive group cannot be symmetrizing, for if this were the case it would be transitive.

A very important consequence is also that *a symmetrizing invariance group forces all invariant matrices to commute with each other*, because the commutator of two symmetric matrices is an antisymmetric, invariant matrix, which must vanish due to the enforced symmetry of all invariant matrices. The centralizer ring of overlap matrices in a symmetrizing group is thus *commutative*.

We shall in the following assume that $G$ is a symmetrizing invariance group, and therefore singly transitive, although many of the general arguments could be carried out even if this were not the case. This means that all magnetic replica values are equal ($m_a = m$) in what follows and the discussion will only focus on the overlap matrix $q$. The assumption that the invariance group enforces symmetry is very strong and permits us to perform a great amount of analysis.

# 7 Double transitivity

The permutation $a \to \pi(a)$ of the $n$ replicas induces a permutation $(a,b) \to (\pi(a), \pi(b))$ of the $n^2$ pairs of replicas. The permutation group $G$ of degree $n$ therefore induces another permutation group of degree $n^2$ which we shall call $2G$. It consists of all diagonal pairs of permutations $(\pi, \pi)$ with $\pi \in G$ and is a subgroup of the direct product $G \otimes G$. Since $2G$ is trivially isomorphic to $G$, its abstract group properties are the same, but the two groups are represented differently in terms of permutations.

Let us denote the orbits of $2G$ by $T_0, T_1, T_2, \ldots, T_I$. Each orbit consists of a set of pairs connected by permutations $\pi \in G$. Since $G$ is assumed to be singly transitive, all diagonal pairs $(a,a)$ form a single orbit (here called $T_0$) by themselves. For $n > 1$ the off-diagonal pairs will be distributed among the other $I$ orbits. If there is only one off-diagonal orbit (I=1) then $G$ is said to be *doubly transitive*. This case is rather uninteresting because it implies that all off-diagonal pairs have the same $q$-value and consequently that the mean field solution is replica symmetric. There are many proper subgroups of the symmetric group that are doubly transitive, for example the alternating group. In order to find a non-trivial solution that breaks replica symmetry we must require that there is more than two non-trivial pair orbits ($I \geq 2$).

Inside each orbit $T_i$ the matrix elements of $q$ are all equal, *i.e.* $q_{ab} = q_i$ for all $(a,b) \in T_i$. Consider for example the symmetrizing invariance group of (10). It has 3 non-trivial pair orbits, namely $T_1 = \{(1,2),(2,1),(3,4),(4,3)\}$, $T_2 = \{(1,3),(3,1),(2,4),(4,2)\}$ and $T_3 = \{(1,4),(4,1),(2,3),(3,2)\}$ and thus the class of solutions (10) is described by 3 non-trivial parameters, $q_1$, $q_2$, and $q_3$. In the case (11) there is only two non-trivial pair orbits.

Let $n_i$ be the number of times that $q_i$ occurs in the first row of $q$. Obviously $n_0 = 1$ and $\sum_i n_i = n$. The transitivity of $G$ then guarantees that each row of the $q$-matrix must be a permutation of the first and thus contain each $q_i$ exactly $n_i$ times. Thus the number of pairs in an orbit is $|T_i| = nn_i$. The assumed transposition symmetry guarantees that if $(a,b) \in T_i$ then $(b,a) \in T_i$, too. Hence the number of pairs $nn_i$ is even in a non-trivial orbit which means that $n$ and $n_i$ cannot both be odd.

The numbers $n_i$ also have another meaning. Let $G_a$ be the *stabilizer* group of $a$ consisting of all those permutations in $G$ that do not affect $a$, *i.e.* $G_a = \{\pi \in G | \pi(a) = a\}$. It then follows that independently of $a$ this group is intransitive with $I+1$ orbits numbered $i = 0, 1, \ldots, I$ of size $n_i$.

## 7.1 Parisi's integral notation

Using these results we may express the term quadratic in the spins in (3) in terms of the non-trivial $q_i$'s

$$\sum_{a \neq b} q_{ab} S_a S_b = \sum_{i>0} q_i \sum_{(a,b)\in T_i} S_a S_b$$

which only depends on the spin-spin products over all pairs in each non-trivial orbit.

One may even introduce a special integral notation first invented by [Parisi (1979a)]. Define the integers $m_i$ by

$$m_i = \sum_{j<i} n_j$$

for $i = 1, 2, \ldots, I+1$ so that $m_1 = 1$ and $m_{I+1} = n$. Then define the piecewise constant function on the interval $1 < x < n$ by

$$q(x) = q_i \quad \text{for } m_i < x < m_{i+1}$$

Using this trick we find for any function $f(q_{ab})$

$$\frac{1}{n} \sum_{a \neq b} f(q_{ab}) = \sum_{i>0} n_i f(q_i) = \int_1^n dx \ f(q(x))$$

which may be used to evaluate the quadratic term in the free energy (2). It has a suggestive 'limit' for $n \to 0$.

Notice that the integral form is exact and arises without any assumptions about the group, except that it is symmetrizing. It does, however, introduce an ordering of the pair orbits reflected in the order on the interval $1 < x < n$. But since there seems to be no natural order of the orbits, the function $q(x)$ is at this point devoid of content. A renumberering of the non-trivial orbits would lead to another function. If the integral notation is to be taken seriously, it is necessary to introduce an order among the pair orbits. We shall see below that the principle of extensibility does exactly that.

# 8 The natural basis

We shall now establish an algebraic basis for dealing with the orbits of pairs $T_i$. For each such orbit we define a characteristic matrix (denoted by the same symbol)

$$T_{i,ab} = \begin{cases} 1, & \text{if } (a,b) \in T_i \\ 0, & \text{otherwise} \end{cases}$$

This matrix is symmetric and invariant. The single transitivity of $G$ implies that there is the same number $n_i$ of 1's in each row of this matrix.

## 8.1 Algebraic properties

From the disjoint nature of the orbits and the transposition symmetry we find that these matrices are orthogonal

$$\frac{1}{n}\mathrm{tr}\,(T_i T_j) = n_i \delta_{ij} \tag{12}$$

and form a natural basis for the centralizer ring

$$q = \sum_{i=0}^{I} q_i T_i \tag{13}$$

where $q_i$ is the value taken by $q$ in $T_i$, also called the coordinate of $q$ along the $i$'th axis.

The fact that all invariant matrices form a ring implies that the sum and product of two invariant matrices is again invariant. Hence the natural basis constitutes a closed algebra

$$T_i T_j = \sum_{k=0}^{I} c_{ij}^k T_k \tag{14}$$

where the coefficients $c_{ij}^k$ are the *structure constants* of the algebra. From this definition and the disjoint nature of the basis matrices it follows that

$$c_{ij}^k = (T_i T_j)_{ab} \quad \text{for all } (a,b) \in T_k \tag{15}$$

which shows that the structure constants are non-negative integers. In fact they are equal to the number of common occurrences of 1's in the $a$'th row of $T_i$ and the $b$'th column of $T_j$, which is never greater than the smallest of $n_i$ and $n_j$. Since the matrix $\sum_j T_j$ has all elements equal to 1, we find the sum rule

$$\sum_{j=0}^{I} c_{ij}^k = n_i \tag{16}$$

From the orthogonality (12) and the fact that all invariant matrices of a symmetrizing invariance group commute we conclude that the symbol

$$d_{ijk} \equiv \frac{1}{n}\mathrm{tr}(T_i T_j T_k) = c_{ij}^k n_k \tag{17}$$

is totally symmetric in all three indices and has the smallest common multiple of $n_i$, $n_j$ and $n_k$ as a factor.

## 8.2 Diagonalisation

The set of commuting symmetric basis matrices $T_i$ of a symmetrizing invariance group may be diagonalized simultaneously. Thus we look for solutions $u$ to

$$T_i u = \lambda_i u$$

for all $i = 0, 1, \dots, I$. One eigenvector may immediately be constructed, namely the one with $u_a = 1$ for all $a$, which has eigenvalues $n_i$, because there is $n_i$ 1's in each row. More generally there will be a set of eigenvalues $\lambda_i^\alpha$ indexed by $\alpha$. Each set may be degenerate and there will be a projection matrix $E_\alpha$ associated with the corresponding degeneracy subspace, satisfying

$$T_i E_\alpha = \lambda_i^\alpha E_\alpha$$

The eigenvalues of course satisfy the algebraic relations of the $T$-matrices

$$\lambda_i^\alpha \lambda_j^\alpha = \sum_k c_{ij}^k \lambda_k^\alpha \tag{18}$$

The projection matrices are orthogonal and complete

$$\begin{aligned} E_\alpha E_\beta &= E_\alpha \delta_{\alpha\beta} \\ \textstyle\sum_\alpha E_\alpha &= 1 \end{aligned}$$

The degeneracy is determined by the dimension of $E_\alpha$

$$D_\alpha \equiv n d_\alpha = \mathrm{tr} E_\alpha \tag{19}$$

where we for later convenience have also defined the fractional dimension $d_\alpha$. The transitivity basis matrices may be expanded in the eigenbasis (using the completeness)

$$T_i = \sum_\alpha \lambda_i^\alpha E_\alpha$$

Conversely, the invariance of the basis $T_i$ implies that a permutation cannot change the eigenvalues but only permute the indices of the degenerate eigenvectors. Hence the degenerate subspaces, and thereby $E_\alpha$ are all invariant matrices that may be expanded in the transitivity basis. Using (12) and (19) the coefficients can be determined

$$E_\alpha = \sum_i \frac{d_\alpha}{n_i} \lambda_i^\alpha T_i$$

The two equations above are reciprocals of each other and the reciprocity implies that the number of eigenvalues is exactly $1 + I$. The eigenvalues themselves form orthogonal systems

$$\left.\begin{array}{rcl} \sum_i \frac{d_\alpha}{n_i} \lambda_i^\alpha \lambda_i^\beta & = & \delta_{\alpha\beta} \\ \sum_\alpha \frac{d_\alpha}{n_i} \lambda_i^\alpha \lambda_j^\alpha & = & \delta_{ij} \end{array}\right\} \tag{20}$$

The special eigenvector, $u_a = 1$, which we shall associate with $\alpha = 0$ and eigenvalues $n_i$, only satisfies the first equation if $d_0 = 1/n$. Since then $D_0 = 1$, this eigenvector is non-degenerate.

The structure constants (17) are completely determined by the eigenvalues

$$d_{ijk} = \sum_\alpha d_\alpha \lambda_i^\alpha \lambda_j^\alpha \lambda_k^\alpha$$

For the symmetrizing invariance group (10) we have $n_1 = n_2 = n_3 = 1$ and the algebra: $T_1^2 = T_2^2 = T_3^2 = 1, T_1 T_2 = T_3$, $T_2 T_3 = T_1$, $T_3 T_1 = T_2$. The eigenvalues become in this case

$$\{\lambda_i^\alpha\} = \begin{array}{c} \alpha\backslash i \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \;\;0\;\; & \;\;1\;\; & \;\;2\;\; & \;\;3\;\; \end{array} \\ \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \end{array}$$

## 8.3 Eigenvalues of the overlap matrix

The eigenvalues of an arbitrary overlap matrix may now be expressed in the form

$$Q_\alpha = \sum_i \lambda_i^\alpha q_i$$

with the inverse relation

$$q_i = \sum_\alpha \frac{d_\alpha}{n_i} \lambda_i^\alpha Q_\alpha$$

By means of the eigenvalues the trace of an arbitrary function of $q$ may be evaluated

$$\frac{1}{n}\mathrm{tr} f(q) = \sum_\alpha d_\alpha f(Q_\alpha) \tag{21}$$

Only one eigenvalue can in the general case be simply expressed in terms of the $q_i$'s, namely

$$Q_0 = \sum_i n_i q_i = q_0 + \int_1^n dx\ q(x)$$

with the inverse relation

$$q_0 = Q_0 + \sum_{\alpha>0} d_\alpha (Q_\alpha - Q_0)$$

written in a way which avoids the explicit appearance of $n$.

# 9 Extensibility

The formulation in terms of integrals seems to suggest that it should be possible to extend the formalism from a given value of $n$ to a higher value $n' > n$ by extending the function $q(x)$ into the region $n < x < n'$ without changing it in the interval $1 < x < n$. This could be done if it were possible to extend the algebra with more basis elements without disturbing the already existing algebraic relations. In the simplest case one might try to add just one extra basis element at a time.

## 9.1 The simply extensible algebra

This is formulated in the following way. We demand that the elements of the natural basis for the centralizer ring of an invariance group can be ordered in such a way that *the members of the truncated set, $T_0, T_1, \ldots, T_i$ form a closed algebra among themselves* for all $i = 0, 1, 2, \ldots, I$. This implies that $c_{ij}^k = 0$ for $k > i, j$, *i.e.* if $k$ is the largest of the three indices. The symmetry (17) under interchange of indices then implies that $c_{ij}^k = 0$, if all indices are different, because one of them must be the largest. The only non-zero structure constants are those where at least two indices are equal. Making use of the sum rule (16) and the symmetry (17) we find $c_{ij}^j = n_i$ for $i < j$, $c_{ii}^j = n_i$ for $j < i$ and $c_{ii}^i = n_i - m_i$. Hence the algebra takes the beautiful form

$$\begin{aligned} T_i T_j &= n_i T_j \qquad \text{for } i < j \\ T_i^2 &= (n_i - m_i) T_i + n_i \textstyle\sum_{j<i} T_j \end{aligned}$$

It is completely defined by the requirement of symmetry and closure of its truncated parts. Since the number $I$ does not appear explicitly, it may also be characterized as being *extensible* in the sense defined above so that new elements may be added to the basis, as long as the algebra closes after each new addition. Notice also that the number of replicas, $n$, does not appear explicitly, but only implicitly through $m_{I+1} = n$. The extensible algebra may in fact be viewed as spanned by an infinite number of commuting basis elements, $T_0, T_1, \ldots$, where each truncated set consisting of the first $I$ closes algebraically on itself.

## 9.2 Eigenvalues

The complete set of eigenvalues for this algebra is given by the $(I+1) \times (I+1)$ matrix

$$\{\lambda_i^\alpha\} = \begin{array}{c} \alpha\backslash i \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ I \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & I \end{array} \\ \begin{pmatrix} n_0 & n_1 & n_2 & n_3 & \cdots & n_I \\ n_0 & -m_1 & 0 & 0 & \cdots & 0 \\ n_0 & n_1 & -m_2 & 0 & \cdots & 0 \\ n_0 & n_1 & n_2 & -m_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n_0 & n_1 & n_2 & n_3 & \cdots & -m_I \end{pmatrix} \end{array} \tag{22}$$

It may be proven by induction on $I$ that these are the correct eigenvalues. This matrix is also extensible and does not refer to $n$ and may be viewed as just the upper left $(I+1) \times (I+1)$ part of an infinite matrix of the indicated form.

The rows are clearly orthogonal as required by (20), which also determines the degeneracies

$$D_\alpha m_\alpha m_{\alpha+1} = n n_\alpha \qquad (\alpha > 0)$$

or

$$d_\alpha = \frac{1}{m_\alpha} - \frac{1}{m_{\alpha+1}} \qquad (\alpha > 0) \tag{23}$$

Since the degeneracy $D_\alpha = n d_\alpha$ must be integer for all $\alpha$ and since $m_1 = 1$, each $m_\alpha$ is a factor of $n$. The extensibility of the scheme requires furthermore that each $m_\alpha$ is a factor of $m_{\alpha+1}$. From this it follows that $m_\alpha$ is also a factor of $n_\alpha$. It is easy to verify that the orthogonality (20) is also fulfilled.

From the eigenvalues (22) we may calculate all the eigenvalues of the $q$-matrix for $\alpha > 0$

$$Q_\alpha = -m_\alpha q_\alpha + \sum_{0 \le i < \alpha} n_i q_i$$

Defining $Q(x) = Q_\alpha$ for $m_\alpha < x < m_{\alpha+1}$ we find for $q_0 = 0$ the relation

$$Q(x) = -x q(x) + \int_1^x dy\ q(y)$$

Finally the trace over an arbitrary function (21) becomes

$$\frac{1}{n}\mathrm{tr} f(q) = f(Q_0) + \int_1^n \frac{dx}{x^2}(f(Q(x)) - f(Q_0))$$

## 9.3 Parisi's prescription

The algebra may be cast into an even nicer form by defining

$$V_i = \sum_{j<i} T_j \quad \text{for } i = 1, 2, \ldots, I+1 \tag{24}$$

It then follows that

$$V_i V_j = m_i V_j \qquad (i \leq j)$$

It is of course not possible to deduce the form of these matrices from their algebra alone, because an orthogonal transformation leaves the algebra unchanged whereas it changes the matrices. But using that all the elements of the $V$'s have to be 0 or 1 (and that $V_{I+1}$ is an $n \times n$ matrix of 1's only), it follows after a suitable renumbering of the replicas that each $V_i$ must be a block diagonal matrix with $n/m_i \times n/m_i$ blocks, where each diagonal block is an $m_i \times m_i$ matrix of 1's and there is 0's everywhere outside the blocks

$$V_i = \begin{array}{c} \\ m_i \\ m_i \\ \vdots \\ m_i \end{array} \!\!\begin{array}{c} \begin{array}{cccc} m_i & m_i & \cdots & m_i \end{array} \\ \begin{pmatrix} 1's & 0's & \cdots & 0's \\ 0's & 1's & \cdots & 0's \\ \vdots & \vdots & \ddots & \vdots \\ 0's & 0's & \cdots & 1's \end{pmatrix} \end{array} \tag{25}$$

Notice that these matrices may be considered to be truncated finite upper-left parts of infinite matrices.

In other words when we number the replicas from 0 to $n-1$

$$V_{i,ab} = \begin{cases} 1, & \text{if } [a/m_i] = [b/m_i] \\ 0, & \text{otherwise} \end{cases}$$

where $[\cdot/\cdot]$ means integer division. From (24) it then follows that

$$T_{i,ab} = \begin{cases} 1, & \text{if } [a/m_i] \neq [b/m_i] \text{ but } [a/m_{i+1}] = [b/m_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

which is the *Parisi prescription* as given in (5). The simplest non-trivial example of an overlap matrix belonging to the Parisi scheme is furnished by (11).

For the symmetrizing invariance groups the Parisi scheme is the only extensible one, leading to a complete ordering of the orbits of double transitivity. Any other scheme must have 'disorder' among the orbits in the sense that there is no way of ordering all the orbits such that the product of $T_i$ with $T_j$ is always expandable in terms of $T_k$ with $k \leq \max(i,j)$.

## 9.4 The replication factors

The Parisi scheme depends on a set of integers $m_i$ with each successive integer being a factor of the preceding one, such that the ratio $r_i = m_{i+1}/m_i$ is an integer greater than 1. This ratio denotes the number of times the generic overlap matrix of dimension $m_i \times m_i$ is replicated along the diagonal to create the generic overlap matrix of dimension $m_{i+1} \times m_{i+1}$.

Without loss of generality one may choose these replication factors to be prime, because the generic overlap matrix for an invariance group with a non-prime replication factor is a special case of the generic overlap matrix for the invariance group with the corresponding prime replication factors. In other words, the Parisi scheme is of the largest generality, *i.e.* contains the largest number of free parameters, when all replication factors are prime. Thus $m_i = \prod_{j<i} r_j$ is simply the resolution of $m_i$ into prime factors. The order of the prime factors is, however, important for the form of the overlap matrix which is demonstrated by the two $6 \times 6$ matrices

$$\left(\begin{array}{cc|cc|cc} 0 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 1 & 0 \end{array}\right)$$

corresponding respectively to $n = 2 \cdot 3$ and $n = 3 \cdot 2$.

Notice also that with this formulation the number of pairs in $T_i$ at any level is even as it should be according to the analysis in section 7, because the product $m_{i+1}n_i$ contains both $r_i$ and $r_i - 1$ as factors. The form of (25) shows that when the sequence of replication factors $r_1, r_2, \ldots$ is given, the Parisi scheme allows the construction of symmetry breaking overlap matrices for all the intermediate $n$-values of the form $km_i$ with $k = 2, \ldots, r_i$ between the levels $i$ and $i+1$. The allowed values for $n$ are thus $n = r_1, 2r_1, \ldots, r_2r_1, 2r_2r_1, \ldots, r_3r_2r_1, \ldots$ and so on, rather than the $m_i$'s only.

In the following we shall see that the replication factors have a natural origin in the structure of permutation groups.

# 10 Imprimitivity

The structure of the Parisi scheme bears a deep relation to the fundamental group theoretical concept of *imprimitivity* which we shall define below. As was mentioned in the beginning of the preceding section, nothing tells us how to order the transitivity basis, but an order seems crucial for the meaningful introduction of a function $q(x)$ which can be extended to higher values of $n$. There are as we shall see good reasons to demand that all the invariance groups should be imprimitive (if at all possible) and this leads naturally to extensible infinite sets of invariance groups similar to the Parisi scheme. As argued above, none of these other schemes can, however, lead to a complete ordering of all the basis elements.

## 10.1 Definition of primitivity

The concept of primitivity arises from the behaviour of permutations when acting on subsets of the replicas. A subset $B \subseteq A$ of replicas is called a *block* if every permutation $\pi \in G$ either maps it completely onto itself ($\pi(B) = B$) or completely into its complement in $A$ ($\pi(B) \cap B = \emptyset$). There are of course trivial blocks consisting of one replica or of all replicas.

A transitive permutation group $G$ is said to be *primitive* if it has no non-trivial blocks. Thus for every non-trivial subset $B \subset A$ there will be a permutation $\pi \in G$ and a pair of replicas $a, b \in B$ such that $\pi(a) \in B$ and $\pi(b) \notin B$. In other words, the image $\pi(B)$ has a non-empty intersection with both $B$ and its complement in $A$. In a primitive group the stabilizer subgroups $G_a$ are maximal, *i.e.* contained in no larger proper subgroup, and conversely if this is the case the group is primitive.

An example of a primitive invariance group of degree 3 is given by (8). This group is, however, not symmetrizing. The only primitive groups of degree 4 are the symmetric group $\mathcal{S}_4$ and the alternating group $\mathcal{A}_4$, both being doubly transitive and therefore leading to replica symmetry. Examples of imprimitive invariance groups of degree 4 are given by (10) and (11). The first example of a non-trivial symmetrizing, primitive group is of degree 5 (and order 10), generated by the permutations $\{(12345), (25)(34)\}$. The invariant overlap matrix is of the form

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{pmatrix} \tag{26}$$

Notice that it is not possible to point to any block structure in this example.

*A doubly transitive group is primitive.* For let us choose an arbitrary non-trivial subset $B \subset A$ and let $a, b \in B$ with $a \neq b$. Then since $G$ is doubly transitive so that all non-diagonal pairs can be mapped to each other, there will be a permutation $\pi$ such that $\pi(a) = a$ and $\pi(b) = c$ where $c \notin B$. Hence $G$ is primitive. Conversely, it follows that *imprimitive groups cannot be doubly transitive.* Imprimitive invariance groups are interesting, because they cannot lead to replica symmetry but necessarily must lead to non-trivial replica symmetry breaking.

The opposite is, however, not true. A primitive group is not necessarily doubly transitive as evidenced by (26) which is also symmetrizing. In order for a primitive group to be doubly transitive something else besides symmetry is needed, for example a subgroup which leaves some replicas invariant and is transitive in the remaining ones.

## 10.2 Basis for imprimitivity

Let us consider an imprimitive group $G$ and let $B$ be a non-trivial block. Let us run through all permutations and form the set of resulting different images $\{A_1, A_2, \cdots, A_r\}$ of $B$, *i.e.* $A_s = \pi_s(B)$ for suitable permutations $\pi_s$. One of the $A_s$'s must be equal to $B$ because the identical permutation maps $B$ onto itself. The transitivity guarantees that every replica must belong to at least one of these subsets such that the set of images covers the set of all replicas. It now follows that the subsets are non-overlapping. For if the same replica was found in $A_s$ and $A_t$ then we would have $\pi_s(a) = \pi_t(b)$ for suitable elements, $a$ and $b$, of $B$. Hence $b = \pi_t^{-1}\pi_s(a)$ would be a permutation mapping an element of $B$ into an element of $B$. Due to $B$ being a block this permutation would map all of $B$ onto itself $B = \pi_t^{-1}\pi_s(B)$, and from this would follow the contradiction $A_t = \pi_t(B) = \pi_s(B) = A_s$. Since every subset must have the same number of elements $m = |A_s|$ it follows that $m$ is a factor of $n$, in fact $n = mr$ with $r \geq 2$. Hence *every transitive group of prime degree is primitive* (this is how the primitivity of the example (26) was inferred).

Any set of non-overlapping subsets $\{A_1, A_2, \cdots, A_r\}$ with the property that every permutation $\pi \in G$ maps the subsets bodily onto each other *i.e.* $\pi(A_s) = A_t$, is called a *basis for imprimitivity*. Here it should be noted that there may be more than one way of choosing a basis for imprimitivity in an imprimitive group (see for example (10) which has 3 different bases of imprimitivity).

Let now $\{A_1, A_2, \cdots, A_r\}$ be a particular basis for imprimitivity of $G$. Each permutation $\pi$ of $G$ acts as a permutation $\rho$ on the $r$ subsets, $\pi(A_s) = A_{\rho(s)}$. This map $\pi \mapsto \rho = \rho(\pi)$ is a homomorphism of $G$ to a permutation group $R$ of degree $r$ which we shall call the *global group*. This group is itself transitive and symmetrizing because of the transitivity and symmetry of $G$. The kernel $K$ of the homomorphism consists of all those permutations that leave all the subsets invariant, *i.e.* $\pi(A_s) = A_s$ for $s = 1, 2, \ldots, r$. It is self-conjugate, *i.e.* $\pi K \pi^{-1} = K$ for all $\pi \in G$, and every coset of the kernel $\pi K$ and only this set of permutations maps to the same global permutation $\rho$ under the homomorphism.

Let $H_s$ be the subgroup consisting of all those permutations in $G$ which leave a particular subset $A_s$ invariant, *i.e.* $\pi(A_s) = A_s$ for $\pi \in H_s$. It easily follows that all these subgroups are conjugate to each other, in fact $\pi H_s \pi^{-1} = H_{\rho(s)}$, and therefore all isomorphic to each other. If $H_s$ is projected on $A_s$, it will act like a permutation group of degree $m$. All the subgroups $H_s$ are therefore homomorphic to the same permutation group $L$ of degree $m$, which we shall call the *local group*. It is evident that the kernel $K$, which is the intersection of all the $H_s$ must map to a self-conjugate subgroup of the direct product of $L$ with itself $r$ times, or $K \subseteq L^{\otimes r}$.

An imprimitive group is thus characterized by two smaller groups: the local group which specifies which permutations are permitted within a single subset of replicas, and the global group which specifies how the subsets of replicas can be permuted among each other. How these two groups collaborate to build up the invariance group is in general quite complicated.

## 10.3 The structure of the overlap matrix

The structure of the invariant matrix corresponding to an imprimitive invariance group is best displayed in block form by means of the $r \times r$ square submatrices $q^{st}$ of size $m \times m$ that the imprimitivity basis defines:

$$\begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & \cdots & A_r \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_r \end{array} & \begin{pmatrix} q^{11} & q^{12} & \cdots & q^{1r} \\ q^{21} & q^{22} & \cdots & q^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ q^{r1} & q^{r2} & \cdots & q^{rr} \end{pmatrix} \end{array}$$

Here we have ordered the replicas globally so that the various subsets of the imprimitivity basis follow each other. Locally we order the replicas in each subset by choosing a fixed set of permutations $\pi_s$ mapping $A_s$ onto $A_{s+1}$ for each $s$, and defining the first element of $A_s$ to map on the first of $A_{s+1}$, the second on the second and so on.

With this ordering it immediately follows that the diagonal block matrices $q^{ss}$ are all identical and invariant under the local group $L$. For the off-diagonal blocks the situation is somewhat more delicate. If the global group $R$ is doubly transitive then all off-diagonal block matrices must be copies of each other apart possibly from local permutations of the rows (or columns). Otherwise, if $R$ is not doubly transitive, each pair orbit will correspond to a differently parametrized matrix.

Imprimitivity gives rise to a partial ordering of the elements of the natural basis of $G$, because some of the basis elements lie entirely within the diagonal blocks and the rest outside. Those belonging to the diagonal blocks must generate a closed algebra among themselves and in this sense be 'smaller' than those belonging to the off-diagonal blocks. If the local group is also imprimitive, its local group in turn defines a set of still 'smaller' basis elements, and so on. Thus, every imprimitive group has an ordering of its natural basis in which subsets of basis elements are nested within each other such that each subset forms a closed algebra by itself.

It will not come as a surprise that the Parisi scheme consists of imprimitive invariance groups nested in this way within each other. The Parisi scheme is, however, as mentioned above, the only one which leads to a full ordering of the whole basis.

## 10.4 Semi-direct products

It is convenient to formalize these considerations by renaming the replicas $(a, s)$ where $a = 1, 2, \ldots, m$ is the local index and $s = 1, 2, \ldots, r$ is the global index. Each permutation in $G$ can be written as $\pi = (\sigma, \rho)$ where $\rho$ is a global permutation of the basis $\{A_1, \ldots, A_r\}$ accompanied by individual local permutations $\sigma = \{\sigma_1, \ldots, \sigma_r\}$ inside each subset, such that $\pi(a, s) = (\sigma_{\rho(s)}(a), \rho(s))$. Defining $(\sigma\sigma')_s = \sigma_s \sigma'_s$ the composition law for the permutations in $G$ takes the form

$$(\sigma, \rho) \cdot (\sigma', \rho') = (\sigma\rho(\sigma'), \rho\rho') \tag{27}$$

where $\rho(\sigma)_s = \sigma_{\rho^{-1}(s)}$ denotes the globally permuted local permutations. The inverse group element is similarly given by

$$(\sigma, \rho)^{-1} = (\rho^{-1}(\sigma^{-1}), \rho^{-1})$$

The kernel $K$ is made up from the special permutations (conserving the global order) of the form $(\sigma, 1)$. If two permutations, $(\sigma, \rho)$ and $(\tau, \rho)$ map to the same global permutation then it follows easily that $\sigma^{-1}\tau$ belongs to the kernel. Hence the range of possible values for the local permutations accompanying $\rho$ is the coset $\sigma K$ where $\sigma$ is any set of local permutations accompanying $\rho$.

In general there will not be permutations (conserving the local order) of the form $(1, \rho)$ for all $\rho$. As we have seen it is always possible to renumber the replicas so that there will be at least $r$ elements of this type (for a specific subset of $\rho$'s). If $(1, \rho)$ belongs to $G$ for all $\rho$ then $R$ is isomorphic to a subgroup of $G$, and $G$ itself will consist of all pairs $(\sigma, \rho)$ with $\sigma \in K$ and $\rho \in R$. The composition law (27) then shows [Gorenstein (1968)] that $G$ is the *semi-direct product* of $K$ and $R$, which shall be written $G = K \odot R$. In this case the local group $L$ will also be obtained by projecting the kernel $K$ onto any of the basis sets $A_s$. For the overlap matrix this means that all off-diagonal blocks belonging to the same global pair orbit must be identical to each other (without additional local permutations).

## 10.5 Building sequences

Imprimitivity guarantees that a symmetrizing invariance group $G$ will give rise to non-trivial replica symmetry breaking. The two derived groups, the local group $L$ of degree $m$ and the global group $R$ of degree $r$ (belonging to a particular basis for imprimitivity) are both symmetrizing invariance groups and $n = mr$. Each of these groups may further be decomposed, and the procedure first stops when both the derived groups are primitive.

Conversely, we may also construct a sequence of permutation groups $G_\ell$ for $\ell = 1, 2, \ldots$ of degree $m_\ell$ such that the local group of $G_{\ell+1}$ is $G_\ell$. The index $\ell$ indicates the level of replica symmetry breaking. At each level we may choose a replication factor $r_\ell$, a kernel subgroup $K_\ell \subseteq G_\ell^{\otimes r_\ell}$ which projects to $G_\ell$, and a global group $R_\ell$, and form the semidirect product $G_{\ell+1} = K_\ell \odot R_\ell$ of degree $m_{\ell+1} = m_\ell r_\ell$.

It is possible to create many sequences of this kind by varying the replication factors $r_\ell$, the way the kernel $K_\ell$ is chosen, and the global group $R_\ell$. There are at least two extreme cases that immediately present themselves (an intermediate scheme is found in [de Dominicis et al (1981)]).

In the first case which is in a sense maximally symmetric one chooses the kernel to be equal to the direct product and the global group to be the full symmetric group, *i.e.* $G_{\ell+1} = G_\ell^{\otimes r_\ell} \odot \mathcal{S}_{r_\ell}$ with $G_1 = \mathcal{S}_1$. This is also called the *wreath product* of $G_\ell$ and $\mathcal{S}_{r_\ell}$ (see for example [Kerber (1971)]). Because of the direct product all matrix elements of $q$ outside the block diagonal must be identical and at each level the natural basis is only extended with a single element. This is therefore the Parisi scheme which has already been fully discussed.

In the second case which is in a sense minimally symmetric the kernel is chosen to consist of similar transformations on each basis set $A_s$, such that $\sigma_1 = \sigma_2 = \cdots = \sigma_r$, or in other words $K = rL$ in the notation of section 7. The kernel simply permutes the replicas in each basis set in the same way instead of independently as in the direct product. If the global group is chosen to be symmetric we get the sequence $G_{\ell+1} = r_\ell G_\ell \otimes \mathcal{S}_{r_\ell}$ with $G_1 = \mathcal{S}_1$ (in this case the semidirect product becomes a direct product because a global permutation does not change the action of the local ones, $\rho(\sigma) = \sigma$). Hence all off-diagonal submatrices of the overlap matrix must be identical and invariant under $G_\ell$ and they will have the same structure as the diagonal matrices but with a different set of parameters. An example of such a matrix is given by (10) and corresponds to the invariance group $2\mathcal{S}_2 \otimes \mathcal{S}_2$. In Appendix A it is shown that this scheme is not acceptable for $n \to 0$.

# 11 Conclusions

We have shown that the theory of permutation groups is important for the analysis of replica symmetry breaking and its consistent extension to the limit of no replicas. In order to derive the Parisi *ansatz* two rather natural conditions must be imposed on the form of the residual invariance group of the replica symmetry breaking mean field solutions.

The first condition is that the invariance group should be *symmetrizing*, which means that it should enforce transposition symmetry of the replica overlap matrix. This matrix is symmetric by definition, but the ring of invariant matrices could in the general case also contain asymmetric matrices. The requirement that the invariance group be symmetrizing makes this ring commutative and forces the invariance group to be transitive. The condition seems natural because the transposition symmetry of the overlap matrix is in fact one of the invariances of this matrix.

The second condition is that the invariance group should be *simply extensible*, which means that it should be possible to order the algebraic basis for the transitive pair orbits such that each truncated set of basis elements closes upon itself. In other words it should be possible to 'peel off' one basis element at a time without disturbing the algebraic relationship between the remaining ones. The ordering is actually a consequence of the imprimitivity of the invariance group. Since the ordering also implies a complete ordering of all the parameters of the overlap matrix, it is possible in a meaningful way to introduce a function of a real variable, a function which in the limit of no replicas becomes the well-known spin glass order parameter. The condition of extensibility is natural from the point of view of generating a model for replica symmetry breaking which allows continuation beyond integral numbers of replicas.

The two conditions, that the invariance group is symmetrizing and simply extensible, have been shown uniquely to characterize the Parisi scheme for replica symmetry breaking. The invariance group of any alternative scheme can therefore either not be symmetrizing or not simply extensible. Intransitive permutation groups cannot be symmetrizing and since it is relatively easy to see that there are fluctuations into intransitive directions near the replica symmetric solutions this possibility merits further investigation. If the group is not simply extensible, it cannot give rise to a complete ordering of the parameters and the meaningfulness of introducing a function of a real variable will be compromised. In one general extensible but not simply extensible case we have shown that the limit of no replicas for the free energy does not exist. Finally it should be mentioned that non-symmetrizing groups can in general not be simply extensible, because the generators of the algebra will come in pairs related by transposition symmetry and this again compromises the use of a function of a real variable as a physical order parameter.

# A An alternative scheme

In section 10 an alternative infinite sequence of invariance groups was constructed by setting $G_{\ell+1} = r_\ell G_\ell \otimes \mathcal{S}_{r_\ell}$ with $G_1 = \mathcal{S}_1$. The overlap matrix at level $\ell + 1$ has of the following structure

$$q^{(\ell+1)} = \begin{pmatrix} q_0^{(\ell)} & q_1^{(\ell)} & \cdots & q_1^{(\ell)} \\ q_1^{(\ell)} & q_0^{(\ell)} & \cdots & q_1^{(\ell)} \\ \vdots & \vdots & & \vdots \\ q_1^{(\ell)} & q_1^{(\ell)} & \cdots & q_0^{(\ell)} \end{pmatrix} \tag{28}$$

where $q_0^{(\ell)}$ and $q_1^{(\ell)}$ are independently parametrized matrices invariant under $G_\ell$. Since the number of parameters satisfies $1 + I^{(\ell+1)} = 2(1 + I^{(\ell)})$ we have $1 + I^{(\ell)} = 2^{\ell-1}$ (in the Parisi scheme we have instead $1 + I^{(\ell)} = \ell$).

Let the natural basis elements at level $\ell$ be denoted $T_i^{(\ell)}$ with index $i = 0, 1, \ldots, I^{(\ell)}$. Due to the binary nature of (28) it is convenient to use a binary notation for the index, writing $i = \sigma_{\ell-1} \ldots \sigma_2\sigma_1$ where the $\sigma$'s are bits with the most significant bit furthest to the left. Let us denote by $D_r$ the $r \times r$ matrix with 1's along the diagonal and 0's outside (the unit matrix) and by $E_r$ the $r \times r$ matrix with 0's along the diagonal and 1's outside. Then the recursion relation for the basis elements can be written as a direct product

$$T_{\sigma i}^{(\ell+1)} = T_i^{(\ell)} \otimes ((1-\sigma)D_{r_\ell} + \sigma E_{r_\ell})$$

From this the recursion relation for the structure constants can be derived

$$c_{\sigma i, \tau j}^{(\ell+1)\phi k} = \left(\delta_{\phi,|\sigma-\tau|} + (r_\ell - 2)\sigma\tau\right) c_{ij}^{(\ell)k}$$

or after expansion

$$c^{(\ell+1)\phi_\ell,\ldots,\phi_1}_{\sigma_\ell,\ldots,\sigma_1;\tau_\ell,\ldots,\tau_1} = \prod_{m=1}^{\ell} \left(\delta_{\phi_m,|\sigma_m-\tau_m|} + (r_m - 2)\sigma_m\tau_m\right)$$

We shall in this appendix limit the study to the simpler case of $r_\ell = 2$ for all $\ell$. In that case the structure constants simplify so that we for all $\ell$ have $\phi_\ell = |\sigma_\ell - \tau_\ell| = \sigma_\ell \wedge \tau_\ell$ where $\wedge$ denotes the 'exclusive or', so that the natural algebra can be written in the form

$$T_i T_j = T_{i\wedge j} \tag{29}$$

where the index on the right hand side is determined by the bitwise application of 'exclusive or'. Notice that there is no use for the level indicator in this form, because 0 bits can be added to the front of the indices without changing the algebra.

It is clear that this system is extensible, level by level. Inside each level there is, however, no way of ordering all the elements of the algebra. The disorder may be observed at level 3 where the two new elements, numbered 2 and 3, satisfy $2 \wedge 3 = 1$. Since $2 \wedge 1 = 3$ and $3 \wedge 1 = 2$ there is no ordering of the two new elements which could lead to a closed algebra consisting of, say, 0, 1 and 2 alone. Below the structure of the invariant matrix is shown at level 4

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Finally it follows from (29) that $T_i^2 = 1$ so that the eigenvalues are all $\pm 1$. Since there are as many parameters as the dimension, *i.e.* $1 + I = n$ the eigenvectors must all be non-degenerate and hence the fractional dimension is $d = \frac{1}{n}$ for all eigenvectors. This seems to prevent the calculation of expressions of the form (21) and consequently the use of this particular scheme in the limit of $n \to 0$.

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